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The Effect of Domain Shape on the Number of Positive Solutions of Certain Nonlinear Equations

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In this paper we discuss how the shape of the domain Ω affects the number of positive solutions of

$$\begin{aligned} -\Delta u &= \lambda f(u) \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega. \end{aligned} \tag{1}$$

Here $f\colon R \to R$ is C^1 and Ω is a bounded domain in R^m where $m > 1$. Usually, we will consider fixed λ though we will sometimes vary λ. We will pay particular attention to the cases where $f(y) = \exp y$ or $f(y) = y^p$ and $\lambda = 1$ (where $1 < p < (m+2)(m-2)^{-1}$). Our main result is that there are *contractible* domains Ω for which our equations with either of these nonlinearities have large numbers of solutions (for fixed λ in the case of the exponential nonlinearity). This contrasts with the case where Ω is a ball where there are fewer solutions (for $m = 2$ in the case of the exponential nonlinearity). Moreover, for each of these solutions, we show that the linearization is invertible. This implies that these solutions continue as solutions of nearby equations. In particular, it follows that in a number of related equations (many of which arise in applications) may have large numbers of solutions on contractible domains. These examples show that the number of solutions is affected by the geometry of the domain and not just its topology. This makes the problem of the number of solutions of our equations appear rather more difficult. In addition, we show that for Eq. (1) with $f(y) = \exp y$ there may be secondary bifurcation of solutions for contractible domains as λ varies and not merely a branch changing direction. We also show that the above two phenomena can occur in star-shaped domains. Here, for fixed λ, we do not obtain large numbers of solutions but only more than for a ball. These results for star-shaped domains are a little surprising because they suggest that the problem of nonuniqueness of positive solutions of (1) for $f(y) = y^p$ with p less than the critical exponent may not be closely related to the problem on nonexistence

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for p equal to the critical exponent. This seems to contradict a number of people's expectations. Our domains Ω are constructed as connected approximations to a finite number of disjoint or touching balls and indeed our techniques are rather insensitive to the type of approximations.

Our results are in fact a little more precise in that we obtain usually an exact count of the number of positive solutions which are not large (and an exact count of all solutions for sublinear problems). On the other hand, we show that for *superlinear* problems the existence of "large" positive solutions is a much more subtle problem which depends on the small scale structure of Ω. We will discuss this briefly.

In addition, for certain nonlinearities (including the ones in Hess [23] and those in some equations in catalysis theory [4]) we show that there can be large numbers of stable positive solutions. Here we mean stable as solutions of a naturally corresponding parabolic equation. Moreover, these stable solutions are less symmetric than the domain Ω.

Some of these results seem to force us to consider more carefully the physical significance of some equations in catalysis theory. We discuss this briefly.

Our main idea can be summarized in the result that, if Ω is an approximation in a rather general sense to a finite union of disjoint balls, then the solutions on Ω which are not large very closely resemble those on the union of the balls.

Last, we obtain some very simple results on the uniqueness of positive solutions on certain highly symmetric domains. This problem seems to require much more work. Indeed, we conjecture that uniqueness holds for $f(y) = y^p$ if Ω is convex and $1 < p < (m+2)(m-2)^{-1}$.

Hale and Vegas [24] and Vegas [38] have studied related problems for Neumann boundary conditions. They study only local problems for Neumann boundary conditions. They study problems where the problem reduces to a two-dimensional bifurcation equation. Moreover, we obtain results for more general domains. Note that Neumann problems (or oblique derivative problems) are rather different from Dirichlet problems. First, for Dirichlet problems, certain extension operators always exist. These only exist for other boundary conditions under additional assumptions. Indeed, under the type of convergence of domains we consider, the eigenvalues of the Laplacian under Neumann boundary conditions need not vary continuously. An example appears in Courant and Hilbert [40, p. 420]. Thus we would not expect our results to hold for other boundary conditions unless we strengthen the convergence requirements on the domain.

Our results for linear problems are related to those of Stummel [35] and Rauch and Taylor [32]. However, the results in these two references do not cover the cases we need. In [28], there is some related work for a modified equation. For linear problems, there was considerable earlier work on the dependence of the eigenvalues upon the domain. We mention in particular Courant and Hilbert [40], Garabedian and Schiffer [41], Necas [44], Babuska and Vyborny [39], and Grigorieff [42]. There is also related work in Stummel [45].

Mignot, Murat, and Puel [43] have also considered, for the Gelfand equation, the main turning point on the solution branch. They consider much more regular variation of the domain (in fact, C^2 small changes) and prove continuity and differentiability of the turning point and obtain a formula for the derivative. We obtain continuity under much weaker assumptions but do not obtain analogues of the other results.

There are a number of interesting two-parameter problems which we have not solved. Here λ becomes large at the same time as Ω degenerates.

In Section 1, we prove the basic results on existence and uniqueness of positive solutions in our special domains. In Section 2, we prove our secondary bifurcation results. In Sections 3 and 4, we discuss applications of physical or mathematical interest. In Section 5, we discuss briefly the possible existence of large solutions in the superlinear case. Finally, in Section 6, we briefly discuss uniqueness.

1. Multiplicity of Solutions on Special Domains

In this section, we discuss positive solutions on the *domain* Ω_n of

$$\begin{aligned} -\Delta u &= f(u) && \text{in } \Omega_n \\ u &= 0 && \text{on } \partial\Omega_n. \end{aligned} \tag{2}$$

Here f is C^1 on R, and there is a $q<(m+2)(m-2)^{-1}$ ($q<\infty$ if $m=2$) such that $|y|^{-q+1}\,|f'(y)|$ is bounded for large y (later, we will show how the second assumption can be largely removed). Note that (2) corresponds to taking $\lambda=1$ in (1). On the Ω_n we assume that there is a finite union of open balls $B=\bigcup_{i=1}^{s} B_i$ where the $\bar{B}_i$ are disjoint and a compact set E of measure zero in R^m such that the following properties hold

(i) given any compact subset K of B, $\Omega_n \supseteq K$ for large n and

(ii) given any open set K_1 containing $E \cup \bigcup_{i=1}^{s} \bar{B}_i$, $\Omega_n \subseteq K_1$ for large n. Our assumptions say that Ω_n is close to B in a rather general sense. Choose a ball $\tilde{B}$ such that $\Omega_n \cup B \subseteq \tilde{B}$ for all n.

Later, we will show that (i) can be weakened still further. Given s and the radii of the B_i, it is easy to construct Ω_n as above such that each Ω_n is contractible. (We simply join suitable B_i by tubes of small width and, if we wish, round off the corners.)

THEOREM 1. (i) *Assume that* $u_0 \in L^\infty(B) \cap \dot{W}^{1,2}(B)$ *is a solution of* (2) (*with* Ω *replaced by* B) *such that the equation*

$$-\Delta h - f'(u_0)\, h = 0$$

has only the trivial solutions in $\dot{W}^{1,2}(B)$. *Suppose that* $\sup\{q, \frac{1}{2}m(q-1)\} < r < 2m(m-2)^{-1}$ ($q < r < \infty$ *if* $m = 2$). *For* n *sufficiently large,* (2) *has a solution* u_n *in* $\dot{W}^{1,2}(\Omega_n)$ *which is close to* u_0 *in* $L^r(\tilde{B})$. *Moreover, this is the only solution in* $\dot{W}^{1,2}(\Omega_n)$ *close to* u_0 *in* $L^r(\tilde{B})$. *In addition, for large* n, *the eigenvalue problems*

$$\begin{aligned} -\Delta h - f'(u_0)\, h &= \lambda h && \text{in } B \\ h &= 0 && \text{on } \partial B \\ -\Delta k - f'(u_n)\, k &= \lambda k && \text{in } \Omega_n \\ k &= 0 && \text{on } \partial \Omega_n \end{aligned} \tag{3}$$

have the same number of negative eigenvalues counting multiplicity and zero is not an eigenvalue for the second problem for large n.

(ii) *Suppose that* K, $\varepsilon > 0$ *and let* S *denote the set of solutions* u *of* (2) *for* $\Omega = B$ *such that* $\|u\|_{1,2} \leqslant K$. *If* n *is large and* u_n *is a solution of* (2) (*for* $\Omega = \Omega_n$) *such that* $\|u_n\|_{1,2} \leqslant K$ *then there is a* $v \in S$ *such that* $\|u_n - v\|_{1,2} \leqslant \varepsilon$. *A similar result holds if we replace the* $\| \ \|_{1,2}$ *by* $\| \ \|_r$.

Remarks. (1) If $\Omega \subseteq B$ and $u \in \dot{W}^{1,2}(\Omega)$, we can extend u to $\dot{W}^{1,2}(\tilde{B})$ by defining it to be zero outside of Ω. Thus we can think of all our functions as being in $\dot{W}^{1,2}(\tilde{B})$. Moreover, when we take the norm of functions, we take the norms over all of $\tilde{B}$.

(2) If (2) has only a finite number of solutions on B each of which is nondegenerate, then, for n large, (2) on Ω_n has exactly this number of solutions except possibly for some large solutions. This result follows by combining the two parts of the theorem.

In a number of places, we will need the following lemma. It is well known but not easy to find in the literature. It follows by combining Theorems 7.10 and 8.15 in Gilbarg and Trudinger [22].

LEMMA 1. *If* $p > 2m(m+2)^{-1}$, $f \in L^p(\Omega)$, $-\Delta u = f$ *in* Ω *and* $u \in \dot{W}^{1,2}(\Omega)$, *then* $\|u\|_{m(p)} \leqslant K_p \|f\|_p$ *where* $m(p) = mp(m-2p)$ *if* $p < \frac{1}{2}m$ *and* $m(p) = \infty$ *if* $p > \frac{1}{2}m$. *Here* $\| \ \|_p$ *denotes the usual norm on* $L^p(\Omega)$. *Moreover, if* $a, b > 0$, *the constants* K_p *are independent of* Ω *for* Ω*'s which satisfy* $a \leqslant$ *measure* $\Omega \leqslant b$.

Proof of Theorem 1. (i) *Step* 1. We prove the *uniqueness*. Assume that u_n, v_n are solutions of (2) for $\Omega = \Omega_n$ such that $u_n \neq v_n$ and $u_n \to u_0$ and

$v_n \to u_0$ in $L^r(\tilde{B})$ as $n \to \infty$. Since the u_n are uniformly bounded in $L^r(\Omega_n)$, Lemma 1 and a simple boot strapping argument shows that they are uniformly bounded in $L^\infty(\tilde{B})$. A similar result holds for the v_n. Now it follows easily that $u_n \to u_0$ in $L^p(\tilde{B})$ for all $p < \infty$.

Let $t_n = (\|u_n - v_n\|_2)^{-1}(u_n - v_n)$. Then $\|t_n\|_2 = 1$ $t_n \in \dot{W}^{1,2}(\Omega_n)$ and

$$-\Delta t_n = f'(\theta_n(x))\, t_n, \tag{4}$$

where $\theta_n(x)$ is between $u_n(x)$ and $v_n(x)$ for all x in Ω_n. Since $u_n \to u_0$ in $L^r(\tilde{B})$, $v_n \to u_0$ in $L^r(\tilde{B})$ and since the u_n and v_n are uniformly bounded in $L^\infty(\tilde{B})$, we easily see that $|f'(\theta_n(x))| \leqslant K_1$ for all n and for all x in Ω_n and that $f'(\theta_n) \to f'(u_0)$ in $L^p(\tilde{B})$ for all $p < \infty$. By (4), it follows that $\|t_n\|_{1,2}$ is uniformly bounded. Hence the t_n (or more precisely their natural extensions) are uniformly bounded in $\dot{W}^{1,2}(\tilde{B})$. Thus, by choosing a subsequence, we can ensure that $t_n \rightharpoonup t$ weakly in $\dot{W}^{1,2}(\tilde{B})$ as $n \to \infty$. Thus, by the Sobolev embedding theorem, $t_n \to t$ strongly in $L^2(\tilde{B})$. Since $\|t_n\|_2 = 1$, $\|t\|_2 = 1$. Thus $t \neq 0$. We will show that $t \in \dot{W}^{1,2}(B)$ and t is a solution of

$$-\Delta w = f'(u_0)\, w \tag{5}$$

in B. Suppose $\phi \in C_0^\infty(B)$. Thus, by our assumptions, if n is large, then $\phi \in C_0^\infty(\Omega_n)$. Hence

$$\int_{\Omega_n} \nabla t_n \nabla \phi = \int_{\Omega_n} f'(\theta_n)\, t_n \phi \tag{6}$$

for n large. Now $t_n \rightharpoonup t$ weakly in $\dot{W}^{1,2}(\tilde{B})$ and $t_n \to t$ strongly in $L^{\hat{p}}(\tilde{B})$ for some $\hat{p} > 2$ (by the Sobolev embedding theorem). Since $f'(\theta_n) \to f'(u_0)$ in $L^p(\tilde{B})$ for all $p < \infty$, it follows that $f'(\theta_n)\, t_n \to f'(u_0)\, t$ in $L^1(\tilde{B})$ as $n \to \infty$. Hence we can pass to the limit in (6) and find that

$$\int_B \nabla t \nabla \phi = \int_B f'(u_0)\, t\phi.$$

(Remember that we can take all the integrations over $\tilde{B}$). Thus t is a weak solution of (4). It remains to prove that $t \in \dot{W}^{1,2}(B)$. Suppose $x_0 \in \tilde{B} \setminus (\bar{B} \cup E)$. If r is small, the closed ball $\bar{B}_r(x_0)$ does not intersect Ω_n for large n. Hence $t_n(x) = 0$ if n is large and $x \in \bar{B}_r(x_0)$. Hence $t(x) = 0$ a.e. on $\bar{B}_r(x_0)$ (since $t_n \to t$ in $L^2(\tilde{B})$). Thus $t(x) = 0$ a.e. on $\tilde{B} \setminus (\bar{B} \cup E)$. Since $\partial \tilde{B} \cup E$ has measure zero, $t = 0$ a.e. on $\tilde{B} \setminus B$. Since B has smooth boundary, $t \in \dot{W}^{1,2}(B)$ cf. [22, p. 168] or Rauch and Taylor [32, p. 29]). Thus t is a weak solution of (5) on B with boundary condition $t = 0$ on ∂B. This contradicts our assumptions.

Step 2. We prove the *existence* of the solution for n large. Define i: $\dot{W}^{1,2}(\tilde{B}) \to \dot{W}^{1,2}(B)$ by $iu = u|_B$. This is a norm decreasing linear map if we put the norm $\|\nabla u\|_2 + \|u\|_2$ on $\dot{W}^{1,2}$. Analogously, we define a map i_n: $\dot{W}^{1,2}(\tilde{B}) \to W^{-1,2}(\Omega_n)$. We define a continuous map L: $W^{-1,2}(B) \to \dot{W}^{1,2}(\tilde{B})$ as follows. If $f \in W^{-1,2}(B)$, $\bar{L}(f)$ is the unique weak solution of $-\Delta u = f$ in B, $u = 0$ on ∂B. $L(f)$ is then defined to be $\bar{L}(f)(x)$ if $x \in B$ and to be zero otherwise. We then define A: $\dot{W}^{1,2}(\tilde{B}) \to \dot{W}^{1,2}(\tilde{B})$ by

$$A(u) = L(f(i(u))).$$

This all makes sense because $u \in L^p(\tilde{B})$ for all $p < 2m(m-2)^{-1}$ and hence $i(u) \in L^p(B)$ for p as above (and thus $f(i(u)) \in W^{-1,2}(B)$). Note that A naturally extends to a completely continuous map of $L^r(\tilde{B})$ into $\dot{W}^{1,2}(\tilde{B})$. A_n is defined analogously (with B replaced by Ω_n). Since the range of A is contained in $\dot{W}^{1,2}(B)$, the fixed points of A are contained in $\dot{W}^{1,2}(B)$ and hence the fixed points of A are the (weak) solutions of (2) (with Ω_n replaced by B). Moreover A is Fréchet differentiable as a map of $L^r(\tilde{B})$ into itself. By Lemma 1, it suffices to prove f is a Fréchet differentiable map of $L^r(\tilde{B})$ into $L^\mu(\tilde{B})$ where $\frac{1}{2}n < \mu < r$. The proof of this follows from Vainberg [36, p. 168]. It is also easy to see that A is completely continuous. Now u_0 is an isolated fixed of A and u_0 has Leray-Schauder index ± 1. This follows since A is Fréchet differentiable at u_0 and since $I - A'(u_0)$ has trivial kernel. The latter statement follows easily from our assumptions once we recall that, since the range of $A'(u_0)$ is contained in $\dot{W}^{1,2}(B)$, the kernel must be contained in $\dot{W}^{1,2}(B)$. Choose $\delta > 0$ such that $u \neq A(u)$ if $u \in L^r(\tilde{B})$, if $\|u - u_0\|_r \leqslant \delta$, and if $u \neq u_0$.

We prove that, if n is large, $y \neq tA(u) + (1-t)\,A_n(u)$ for $0 \leqslant t \leqslant 1$, $u \in \dot{W}^{1,2}(\tilde{B})$, and $\|u - u_0\|_r = \delta$. Suppose not. Then there exist $t_n \in [0, 1]$ and $u_n \in \dot{W}^{1,2}(\tilde{B})$ such that $\|u_n - u_0\|_r = \delta$ and

$$u_n = t_n A(u_n) + (1 - t_n)\,A_n(u_n). \tag{7}$$

By Lemma 1 and a simple bootstrapping argument, we see the u_n are uniformly bounded in $L^\infty(\tilde{B}) \cap \dot{W}^{1,2}(\tilde{B})$. Since $\{u_n\}$ is bounded in $\dot{W}^{1,2}(\tilde{B})$, we can choose a subsequence converging weakly to $v \in \dot{W}^{1,2}(\tilde{B})$ and strongly in $L^r(\tilde{B})$. Hence $\|v - u_0\|_r = \delta$. Suppose T is an open ball with $\bar{T} \subseteq B$. Then $T \subseteq B$ and $T \subseteq \Omega_n$ for large n. Thus, by taking the Laplacian of (7) on T for n large we see that

$$-\Delta u_n(x) = f(u_n(x)) \tag{8}$$

if $x \in T$ and n is large. (Note that $-\Delta A(u) = f(u)$ on B and $-\Delta A_n(u) = f(u)$ on Ω_n). Multiplying (7) by $\phi \in C_0^\infty(T)$ and passing to the limit we see that

$$\int_T \nabla v\, \nabla\phi = \int_T f(v)\, \phi.$$

Thus $-\Delta v = f(v)$ on T. Hence $-\Delta v = f(v)$ on B. Next we prove that $v \in \dot{W}^{1,2}(B)$. Suppose K_1 is an open neighbourhood of $\bar{B}$. Since $\Omega_n \subseteq K_1$ for n large and $B \subseteq K_1$ and since the $tA(u)(x) + (1-t)A_n(u)(x) = 0$ if $x \notin \bar{B} \cup \Omega_n$ (by the definitions of A and A_n) we see that $u_n(x) = 0$ if n is large and $x \notin K_1$. Hence, passing to the limit, we see that $v(x) = 0$ if $x \notin K_1$. Hence $v(x) = 0$ a.e. on $\tilde{B} \backslash (\bar{B} \cup E)$. Since $E \cup \partial B$ has measure zero, $v(x) = 0$ a.e. on $\tilde{B} \backslash B$. As before, it follows that $v \in \dot{W}^{1,2}(B)$. Hence $v = A(v)$ (by the definition of A). Thus contradicts the results of the previous paragraph and hence our claim at the start of the paragraph is proved.

We now complete the proof of the existence of u_n for n large. Let $\mathbf{B}$ denote the ball of radius δ and centre u_0 in $L^r(\tilde{B})$. By the result of the previous paragraph,

$$\deg(I - A_n, 0, \mathbf{B}) = \deg(I - A, 0, \mathbf{B}). \tag{9}$$

Here we are thinking of A and A_n as maps on $L^r(\tilde{B})$. Since u_0 is the only fixed point of A in $\mathbf{B}$, it follows that the right hand side of (9) is $\operatorname{index}(I - A, u_0) = \pm 1$ since A is Fréchet differentiable on $L^r(\tilde{B})$ and $I - A'(u_0)$ is invertible. Thus $\deg(I - A_n, 0, \mathbf{B}) \neq 0$ for large n and hence there exists u_n near u_0 in $L^r(\tilde{B})$ such that $u_n = A_n(u_n)$. Hence the existence follows.

Step 3: *Linearizations.* First, a very similar (but slightly easier) argument to that in the proof of Step 1 shows that $-\Delta - f'(u_n)\, I$ (on Ω_n with Dirichlet boundary conditions) is invertible for large n. This proves one claim. Next, note that there is a $K > 0$ independent on n such that any eigenvalue λ of (3) satisfies $\lambda \geqslant -K$. This follows easily from the variational characterization of the first eigenvalue once we recall that the first eigenvalue of $-\Delta$ on Ω_n is nonnegative and that $f'(u_n(x)) \leqslant \sup\{f'(s) : |s| \leqslant t\}$ on Ω_n where $|u_n(x)| \leqslant t$ for all $x \in \Omega_n$ and all n. (By the argument in Step 2 of the proof, $t < \infty$.) Let k denote the number of negative eigenvalues of $-\Delta - f'(u_0)\, I$ (on B with Dirichlet boundary conditions). k_n is defined analogously (with B replaced by Ω_n). To prove that $k_n \geqslant k$ for large n, we argue as follows. By the definition of k, there is a k-dimensional subspace T of $\dot{W}^{1,2}(B)$ such that $H(v) = \int_B \frac{1}{2} (\nabla v)^2 - \frac{1}{2} f'(u_0)\, v^2 < 0$ on $T \backslash \{0\}$. (T is the subspace spanned by the eigenvectors corresponding to negative eigenvalues.) We can approximate each element of a basis for T (in $\dot{W}^{1,2}(B)$) by elements of $C_0^\infty(B)$. Thus, by continuity, we obtain a k-dimensional subspace $\tilde{T}$ of $C_0^\infty(B)$ such that $H(v) < 0$ if $v \in \tilde{T}$, $\|v\|_{1,2} = 1$. Hence $H(v) < 0$ on $\tilde{T} \backslash \{0\}$. Since $\tilde{T}$ is finite-dimensional, there is a compact subset K_1 of B such that $\operatorname{supp} v \subseteq K_1$ for all $v \in \tilde{T}$. Hence, if n is large, $\operatorname{supp} v \subseteq \Omega_n$ for n large. Thus we can think of $\tilde{T}$ as a subspace of $\dot{W}^{1,2}(\Omega_n)$ for n large. By the dominated convergence theorem and the compactness of the unit ball in $\tilde{T}$,

$$\int_{\Omega_n} f'(u_n)\, v^2 \to \int_B f'(u_0)\, v^2 \text{ as } n \to \infty \text{ uniformly for } v \in \tilde{T},\ \|v\|_{1,2} = 1.$$

Hence we see that, if n is large,

$$H_n(v) \equiv \int_{\Omega_n} \frac{1}{2} |\nabla v|^2 - \frac{1}{2} f'(u_n) v^2 < 0$$

for $v \in \tilde{T}$, $\|v\|_{1,2} = 1$. Thus H_n is negative on a k-dimensional subspace of $\dot{W}^{1,2}(\Omega_n)$ (except at zero) and hence $k_n \geqslant k$ for large n (by the variational characterization of eigenvalues). Suppose now that our result is false. Then, by choosing a subsequence if necessary, we can assume that $k_n \geqslant k+1$ for all large n. Thus for $n \geqslant n_0$, the eigenvalue problem (3) has eigenfunctions $\{v_{n,j}\}_{j=1}^{k+1}$ corresponding to negative eigenvalues $\lambda_{n,j}$ and such that $\{v_{n,j}\}_{j=1}^{k+1}$ is orthogonal. Without loss of generality, $\|v_{n,j}\|_2 = 1$ for $n \geqslant n_0$, $1 \leqslant j \leqslant k+1$. Since

$$\int_{\Omega_n} |\nabla v_{n,j}|^2 = \int_{\Omega_n} (f'(u_n) v_{n,j}^2 + \lambda_{n,j} v_{n,j}^2)$$

and since the $\lambda_{n,j}$ are bounded (as they are negative and bounded from below), we see that $\{v_{n,j}\}$ is bounded in $\dot{W}^{1,2}(\tilde{B})$. Thus, by choosing subsequences if necessary, we can assume that $\{v_{n,j}\}_{n=n_0}^{\infty}$ converges weakly to v_j in $\dot{W}^{1,2}(\tilde{B})$ and $\lambda_{n,j} \to \lambda_j$ as $n \to \infty$ for $1 \leqslant j \leqslant k+1$. Hence $v_{n,j}$ converges strongly in $L^2(\tilde{B})$. Thus $\|v_j\|_2 = 2$ and $\{v_j\}_{j=1}^{k+1}$ is orthogonal. By using a similar limit argument to that in the proof of Step 1, $v_j \in \dot{W}^{1,2}(B)$ and v_j is an eigenvector of $-\Delta - f'(u_0) I$ corresponding to the eigenvalue λ_j. Since $\lambda_{n,j} < 0$, $\lambda_j < 0$. (Note that $\lambda_j \neq 0$ by the first part of the proof of Step 3.) Hence there are at least $k+1$ eigenvalues of $-\Delta - f'(u_0) I$ on B corresponding to negative eigenvalues. This is impossible since this contradicts the definition of k. This completes the proof of (i).

(ii) We prove the result for $\| \|_{1,2}$. The proof for $\| \|_r$ is similar but easier. We prove that if u_n are solutions of (2) for $\Omega = \Omega_n$ such that $\|u_n\|_{1,2} \leqslant K$, then a subsequence converges *strongly* in $\dot{W}^{1,2}(\tilde{B})$ to a solution v of (2) for $\Omega = B$. The result follows easily from this and s simple compactness argument. Once we note that a sequence u_n as above must be bounded in $L^r(B)$, the *weak* convergence in $\dot{W}^{1,2}(\tilde{B})$ to a solution v on B follows easily from our earlier arguments. To prove strong convergence, note that

$$\begin{aligned}\int_{\tilde{B}} |\nabla v_n|^2 = \int_{\Omega_n} |\nabla v_n|^2 = \int_{\Omega_n} f(v_n) v_n = \int_{\tilde{B}} f(v_n) v_n \\ \to \int_{\tilde{B}} f(v) v \\ = \int_{\tilde{B}} |\nabla v|^2.\end{aligned}$$

Here we used the compactness of the embedding in the Sobolev embedding theorem for an exponent less than the critical exponent. Note that the proof of the last inequality is the same as the proof of the corresponding equality for v_n. Hence $v_n \to v$ weakly in $\dot{W}^{1,2}(\tilde{B})$ and $\|v_n\|_{1,2} \to \|v\|_{1,2}$. Thus $v_n \to v$ strongly in $\dot{W}^{1,2}(\tilde{B})$ as required.

Remarks. 1. First, we need not make a growth assumption on f' but only the corresponding growth assumption on f. To see this, note that our arguments imply that we have a uniform bound in the L^∞ norm for solutions u which satisfy $\|u\|_{1,2} \leqslant K$ or u is near u_0 in $L^r(\tilde{B})$. Here the bound is independent of n and of f provided that $|f(y)| \leqslant C_1 + C_2|y|^q$ on R. (The bound does depend on C_1 and C_2.) Thus we can modify f for $|y|$ large so as not to affect the solutions we are studying (because the growth rate on f is unchanged) but now f' is bounded on R. This device can also be used to improve our growth assumptions on f when $m = 2$. It suffices to assume that $|f(y)| \leqslant C + \exp C|y|^\alpha$ for some $\alpha < 2$ and $C > 0$. Here we replace closeness in the L^r norm by closeness in the Orlicz norm $L_{\tilde{\phi}}$ where $\tilde{\phi}(t) = \exp|t|^\beta$ with $\alpha < \beta < 2$, and boundedness in $\dot{W}^{1,2}(\tilde{B})$ by boundedness in $L_{\tilde{\phi}}(\tilde{B})$. This follows easily from our truncation argument once we note that, if u is bounded in $L_{\tilde{\phi}}$, $f(u)$ is bounded in $L^p(\tilde{B})$ for $p > 1$ and hence, if u is a solution of (2), u is bounded in $L^\infty(\tilde{B})$. We can in fact use the same idea to prove a theorem under no growth assumption on f at all. We choose a C^2 even convex function ϕ such that $|f(t)|/\phi(t) \to 0$ as $|t| \to \infty$. Then Theorem 1 holds if we replace the L^r norm by the L_ϕ norm and always look at solutions in L^∞. The only other difference is that we must now delete the first statement of Theorem 1(ii).

2. The proof does not use that B is a finite union of balls with disjoint closures. All we need of B is that B is open, that ∂B has zero measure, and that, if $u \in \dot{W}^{1,2}(\tilde{B})$ and $u(x) = 0$ if $x \notin B$, then $u \in \dot{W}^{1,2}(B)$. In fact, the only reason for choosing a finite union of balls is that, it is usually much easier to understand (2) on a ball than on more complicated domains. Moreover, we can allow the case where B is a union of disjoint open balls B_i, $i = 1, ..., s$, but the $\bar{B}_i$ are not disjoint. (Necessarily only two of the $\bar{B}_i$ can intersect at any one point and there are only a finite number of such points $\{x_i\}_{i=1}^p$.) To prove that our argument is still valid, we only have to prove that if $u \in \dot{W}^{1,2}(\tilde{B})$ and $u(x) = 0$ for $x \notin B$ then $u \in \dot{W}^{1,2}(B)$. We use the notion of capacity (cf. Kinderlehrer and Stampacchia [29]). Since $\{x_1\}$ has zero capacity in $\tilde{B}$, there exists $\phi_n \in C_0^\infty(\tilde{B})$ such that $\phi_n = 1$ in a neighbourhood of x_1, $\int_{\tilde{B}} |\nabla\phi_n|^2 \leqslant 1/n$ and $0 \leqslant \phi_n \leqslant 1$ in $\tilde{B}$ [29]. Note that we can assume u is bounded on $\tilde{B}$ since $f_n(u) \to u$ in $\dot{W}^{1,2}(\tilde{B})$ as $n \to \infty$ (where f_n is bounded on R, $f_n(0) = 0$, f_n is C^1, $|f_n'(y)| \leqslant 1$ on R, $f_n'(y) \to 1$ uniformly on compact sets). Note that $f_n(u)$ is also zero outside of B. Now it is easy to show that $(1 - \phi_n(x))\, u \to u$ in $\dot{W}^{1,2}(\tilde{B})$ as $n \to \infty$. Thus, we see

that it suffices to assume that u vanishes in a neighbourhood of x_1. Repeating the argument, we see that it suffices to assume that u vanishes in a neighbourhood of each of the points where the spheres touch. We can now complete the argument by approximating $u|_{B_i}$ for each i. We can do this because the supports are now a positive distance apart. The same argument (and a much easier converse argument) implies that $u \in \dot{W}^{1,2}(B)$ if and only if $u|_{B_i} \in \dot{W}^{1,2}(B_i)$ for each i. It follows easily from this that u is a weak solution of (2) on B if and only if u is a weak solution on each B_i. Note that this is valid even if the spheres touch. The main reason for the interest in allowing touching spheres is that we can do this (at least for two touching balls) and keep each Ω_n *star shaped.* (For example, we could choose each Ω_n to be the union of balls with a small overlap.) Note that such a set is star shaped about any point of their intersection.

With a little more care, one can do this so that $\partial\Omega_n$ has smooth boundary and Ω_n has an $O(m-1)\times Z_2$ symmetry.

3. One can improve our results on how u_n converges to u_0. By applying Harnack's inequality to $u_n - u_0$, we see that, if K is compact subset of B, then $u_n \to u_0$ uniformly on K. Moreover, one can use barriers (cf. [22, p. 198]) to prove that $u_n \to u_0$ uniformly near "nice" parts of the boundary. However, in general, we do not expect u_n to converge uniformly to u_0. To see this, one chooses domains Ω_n converging in our sense to Ω_0 where Ω_n is reasonably smooth but Ω_0 is irregular enough such that the solution of $-\Delta u = 1$ in Ω_0 with Dirichlet boundary conditions is not continuous on all of $\partial\Omega_0$. However, u_n is continuous so that u_n can not converge uniformly to u_0. On the other hand, barriers can be used to prove uniform convergence for the simplest dumbell problem.

4. The existence of the u_n is ensured if we delete our invertibility assumption but assume that index $(A, u_0) \neq 0$. Note that one can use the product theorem for the degree to show that this index is the product of indices on each B_i. Last, condition (i) on the Ω_n can be weakened to the requirement that there is a compact set K_2 of B of capacity zero so that (i) holds for B replaced by $B \backslash K_2$. Moreover, if f is Lipschitz on R and C^1 except at z, our argument is still valid provided that $\{x \in B: u_0(x) = z\}$ has measure zero.

5. Our proof of Step 3 becomes easier to understand if we realize that our argument can be used to prove that $A_n(u_n) \to A(u)$ in $L^r(\tilde{B})$ if $u_n \to u$ in $L^r(\tilde{B})$. We did not state it in this form because the fact that we only need results of this type for u_n uniformly bounded in $L^\infty(\tilde{B})$ is convenient when one does generalizations to cases of weaker growth conditions on f. Note that the proof of the result above for A_n easily redues to proving an analogous convergence result for $(-\Delta)^{-1}$.

If u_0 is nonnegative on B, we often want to know whether u_n is nonnegative on Ω_n for large n. If f is nonnegative on R, this is obvious but we often want to handle more general situations. If $f(0) \geqslant 0$, the weak Harnack inequality implies that if B_i is a component of B and u_0 is nonnegative on B, then either $u_0(x) > 0$ for every $x \in B_i$ or $u_0(x) = 0$ for every $x \in B_i$. Moreover, the last possibility can only hold if $f(0) = 0$. If u_0 vanishes identically on B and if our invertibility conditions holds, then u_n must be the solution vanishing identically by uniqueness. Thus the result is obvious in this case. Let B^s denote the union of the components of B on which u_0 vanishes identically and let λ_s denote the first (weak) eigenvalue of

$$\begin{aligned} -\Delta h &= \lambda h \qquad \text{in } B^s \\ h &= 0 \qquad \text{on } \partial B^s. \end{aligned}$$

THEOREM 2. *Assume that the conditions of Theorem* 1 *hold, that* $f(0) \geqslant 0$, *and* u_0 *is nonnegative on* B *but does not vanish identically.*

(i) *If* $f(0) > 0$ *or* $u_0(x) > 0$ *on* B *or* $f'(0) < \lambda_s$, *then* u_n *is nonnegative on* Ω_n *for all large* n.

(ii) *If* $f(0) = 0$ *and* $f'(0) > \lambda_s$, *then* u_n *changes sign on* Ω_n *for all large* n.

Remark. Our assumptions ensure that $f'(0) \neq \lambda_s$ if $B^s \neq \varnothing$.

We summarize a special case of the main result in [14] because we use it essentially below. Assume that K is a cone in a Banach space E such that $K - K$ is dense in E, $\tilde{A}: E \to E$ is compact, and C^1, $\tilde{A}(K) \subseteq K$ and x_0 is a fixed point of $\tilde{A}$ in K such that $I - \tilde{A}'(x_0)$ is invertible. Let $W = \{s \in E: u_0 + \varepsilon s \in K$ for some small positive $\varepsilon\}$ and let $S = \bar{W} \cap (-\bar{W})$. It is shown in [14] that S is a closed subpace of E and $\tilde{A}(x_0)$ maps S into itself. Let C denote the map induced by $\tilde{A}'(x_0)$ on E/S. If the spectral radius $r(C) > 1$, then $\operatorname{index}_K(\tilde{A}, u_0) = 0$. Otherwise, $\operatorname{index}_K(\tilde{A}, u_0) = \pm 1$. Here index_K denotes the index of the solution relative to the cone K. If, as in our case, S has a natural complement $\tilde{S}$ which is $\tilde{A}'(u_0)$ invariant, then $r(C) = r(A'(u_0)|_{\tilde{S}})$. u_0 is said to be demi-interior if $\bar{W} = E$.

Proof of Theorem 2. First assume that $f(y) \geqslant 0$ for $y \geqslant 0$. Let K denote the natural cone in $L^r(\tilde{B})$. We will use the degree on the cone K as in [14]. Note that A and A_n map K into itself as does the homotopy we constructed joining them.

We first prove that, if $\operatorname{index}_K(A, u_0) \neq 0$, then u_n is nonnegative on Ω_n for large n while, if the index is zero, then u_n changes sign on Ω_n for all large n. If $\mathbf{B}$ is a suitable small neighbourhood of u_0 in $L^r(\tilde{B})$, the same proof as in the existence part of Theorem 1(i) shows that

$$\deg_K(A_n, 0, \mathbf{B} \cap K) = \deg_K(A, 0, \mathbf{B} \cap K) = \operatorname{index}_K(A, u_0) \tag{10}$$

for large n. Hence, if $\operatorname{index}_K(A, u_0) \neq 0$, then there exists a solution of (2) (for $\Omega = \Omega_n$) in $\mathbf{B} \cap K$. In particular, there is a nonnegative solution of (2) (for $\Omega = \Omega_n$) near u_0. The result follows in this case from the uniqueness in Theorem 1(i). Conversely, assume that $\operatorname{index}_K(A, u_0) = 0$. Thus for n large, $\deg_K(A_n, 0, \mathbf{B} \cap K) = 0$ (by (10)). If u_n is nonnegative on Ω_n, this and the uniqueness in Theorem 1(i) imply that $\operatorname{index}_K(A_n, u_n) = 0$. However, since Ω_n is connected, our earlier arguments imply that $u_n(x) > 0$ on Ω_n and thus we easily see that u_n is demi-interior to K (in our earlier sense). Thus we see, by our earlier comments that u_n has index ± 1 in K. Hence we have a contradiction and thus u_n must change sign in Ω_n.

To complete the proof, we must evaluate $\operatorname{index}_K(A, u_0)$. If $f(0) > 0$ or more generally $u_0(x) > 0$ on B, the same argument as in the last part of the previous paragraph shows that $\operatorname{index}_K(A, u_0) = \pm 1$ and the result follows as in the previous paragraph. In the general case, to evaluate the index, we use our comments after the statement of Theorem 2. By an elementary calculation, one finds that $\bar{W} = \{u \in L^r(B): u(x) = 0 \text{ a.e. on } B^s\}$. As earlier, this is $A'(u_0)$ invariant. Here we use that $u_0(x) > 0$ on $B \backslash B^s$. Now the orthogonal complement to $\bar{W}$ is $M = \{u \in L^r(B): u(x) = 0 \text{ on } B \backslash B^s\}$. One easily sees that M is $A'(u_0)$ invariant. Now $\operatorname{index}_K(A, u_0) \neq 0 \Leftrightarrow r(A'(u_0)|_M) < 1$. However, on M, $A'(u_0) = f'(0)(-\Delta)^{-1}$ (where we mean the Laplacian on B^s). Thus we see that $\operatorname{index}_K(A, u_0) \neq 0$ if and only if $f'(0) < \lambda_s$, as required. (Note that our other conditions ensure that $f'(0) \geqslant 0$ if $f(0) = 0$.)

The condition that $f(y) \geqslant 0$ for $y \geqslant 0$ can be removed by using our earlier idea to modify f so that $f(y) \geqslant 0$ for large y and then choose $\alpha \geqslant 0$ such that $f(y) + \alpha y \geqslant 0$ for $y \geqslant 0$. We then define A and A_n by using $(-\Delta + \alpha I)^{-1}(f(u) + \alpha u)$ instead of $(-\Delta)^{-1} f(u)$. The rest of the argument is essentially unchanged.

Remarks. We can clearly prove an analogue of Theorem 1(ii) for nonnegative solutions. Moreover, one can prove analogues of all the remarks after the proof of Theorem 1. If $f(0) > 0$ or if $f'(0) = 0$ one can give an alternative proof of Theorem 2 by using a modified f with the property that $f(x) \geqslant 0$ on R. Note that λ_s is the first eigenvalue of $-\Delta$ on the largest of the balls which make up B^s. Our methods can clearly be used for more general second-order differential operators and systems. Note, however, that, if we lose self-adjointness we can only prove much weaker versions of our result on the number of negative eigenvalues of the linearizations, and that one can only prove Theorem 2 for systems with appropriate positivity properties. Moreover, for *some* systems with positivity one can still obtain stability results by considering the principal eigenvalue.

2. Bifurcation

In this section, we discuss the case where f depends smoothly on a scalar parameter λ. We will only consider the case where $f(x, \lambda) = \lambda f(x)$ though our methods work much more generally. Thus we consider the equation

$$\begin{aligned} -\Delta u &= \lambda f(u) \qquad &&\text{in } \Omega \\ u &= 0 &&\text{on } \partial\Omega. \end{aligned} \tag{11}$$

We assume that Ω_n satisfy the assumptions of Section 1. First, suppose that $u_0(\lambda)$ is a solution of (11) (for $\Omega = B$) in $L^\infty(B) \cap \dot{W}^{1,2}(B)$ for $\lambda_1 \leqq \lambda \leqq \lambda_2$, such that $u_0(\lambda)$ depends continuously on λ and the linearization of (11) at $u_0(\lambda)$ is invertible. Then there is an $\varepsilon > 0$ such that for each sufficiently large n, (11) (for $\Omega = \Omega_n$) has a unique solution $u_n(\lambda)$ in $\{u \in L^r(\tilde{B}): \|u - u_0(\lambda)\|_r \leqq \varepsilon\}$. Here r is as in Section 1. Moreover, $u_n(\lambda)$ depends continuously on λ and the corresponding linearization is invertible. This follows by examining the proofs in Section 1 to check that everything can be done uniformly in λ. Here we are assuming the $|f(y)| \leqq C_1 + C_2|y|^q$ on R for some $q < (m+2)(m-2)^{-1}$. Otherwise we must replace $\| \ \|_r$ by an appropriate Orlicz norm. This applies to all the results of this section.

We next consider the case of a simple bifurcation in the sense of Crandall and Rabinowitz [7]. Thus we assume that (u_0, λ_0) is a solution of (11) for $\Omega = B$ such that (u_0, λ_0) is not nondegenerate but the mapping $(h, t) \to -\Delta h - \lambda_0 f'(u_0)\, h - tf(u_0)$ is onto as a map of $\dot{W}^{1,2}(B) \times R \to W^{-1,2}(B)$. One easily sees that this is equivalent to assuming that the linearization of (11) at u_0 (for $\lambda = \lambda_0$) has a one-dimensional kernel spanned by h_0 and $\int_B f(u_0)\, h_0 \neq 0$. Note that there is some freedom in the choice of spaces. For example, we could replace $\dot{W}^{1,2}(B)$ by $T = \{u \in \dot{W}^{1,2}(B): \Delta u \in L^\infty(B)\}$ and $W^{-1,2}(B)$ by $L^\infty(B)$. These are the spaces we will mainly use. Here we use the norm $\|\Delta u\|_\infty$ on T (for $u \in T$). For *simplicity*, we will also assume that $f \in C^2(R)$ and $\int_B f''(u_0)\, h^3 \neq 0$. This seems to be the most important case and is the case where the results are tidiest. (In the general case, the situation is a little like that in the perturbation results of Section 1 of [9]). Note, that, under our growth assumption on f, solutions bounded in $L^r(\Omega_n)$ are bounded in $L^\infty(\Omega_n)$. Moreover, using this and the theory of linear equations, one sees that solutions close in $L^r(\Omega_n)$ are close in $L^\infty(\Omega_n)$ (where we are considering solutions on the same domain). Here, as in the previous section, the constants are independent of n. Thus, we see that we will not lose anything by working in $L^\infty(\Omega_n)$.

We first consider the problem on B. By our earlier remarks and Theorem 2.1 in [1], we see that there is an $\varepsilon > 0$, a C^2 functional

$\phi\colon(-\varepsilon,\varepsilon)\to R$, and a C^2 function $\psi\colon(-\varepsilon,\varepsilon)\to R_0=\{u\in T\colon\int_B uh_0=0\}$ such that $\phi(0)=0$, $\psi(0)=0$, $\psi'(0)=0$, $\phi'(0)=0$, $\phi^2(0)\neq 0$ and solutions of (1) close to (u_0,λ_0) in $L^\infty(\Omega)\times R$ are $\{(u_0+\alpha h_0+\psi(\alpha),\ \lambda_0+\phi(\alpha))\colon \alpha\in(-\varepsilon,\varepsilon)\}$. Note that solutions close in $L^\infty(B)$ are close in T. We prove that a similar result holds on Ω_n (and holds uniformly in n).

THEOREM 3. *Assume that the conditions of Section* 1 *on the* Ω_n *hold, that* f *is* C^2 *on* R, *that* (u_0,λ_0) *is a simple bifurcation point in the above sense, and that* $\int_B f''(u_0)\,h^3\neq 0$ *where* h *spans the kernel of* $-\Delta-\lambda_0 f'(u_0)\,I$. *Then there exists* (u_n,λ_n) *close to* (u_0,λ_n) *in* $L^r(\tilde{B})\times R$ *which is a simple bifurcation point in our earlier sense. In addition, there are* $\varepsilon>0$, C^2 *functionals* $\phi_n\colon(-\varepsilon,\varepsilon)\to R$, $\psi_n\colon(-\varepsilon,\varepsilon)\to R_n\equiv\{u\in T_n\colon\int_{\Omega_n} uh_n=0\}$ *such that* $\phi_n(0)=0$, $\psi_n(0)=0$, $\psi_n'(0)=0$, $\phi_n(0)=0$, $\phi_n^2(0)\neq 0$ *and the solutions close to* (u_n,λ_n) (*for* $\Omega=\Omega_n$) *are* $\{(u_n+\alpha h_n+\psi_n(\alpha),\ \lambda_n+\phi_n(\alpha))\colon\alpha\in(-\varepsilon,\varepsilon)\}$. *Here* h_n *spans the kernel of* $-\Delta-\lambda_n f'(u_n)\,I$, $T_n=\{u\in\dot{W}^{1,2}(\Omega_n)\colon \Delta u\in L^\infty(\Omega_n)\}$ *and the size of the neighbourhood is independent of* n. *Moreover* $\phi_n\to\phi$ *and* $\psi_n\to\psi$ *uniformly on* $(-\varepsilon,\varepsilon)$ *as* $n\to\infty$.

Remarks. As before, one can remove the growth condition on f by replacing $\|\ \|_r$ by a suitable Orlicz norm. Second, one can prove that $\phi_n''(\alpha)\neq 0$ for $\alpha\in(-\varepsilon,\varepsilon)$ (and thus there is only the one turning point). The idea to prove this is to differentiate the equation for (ψ_n,ϕ_n) to obtain a formula for $\phi_n''(\alpha)$ and then use our earlier ideas to prove that $\phi_n''(\alpha)\to\phi''(\alpha)$ uniformly on $(-\varepsilon,\varepsilon)$. Third, by our earlier comments, the above solutions are the only ones close to (u_0,λ_0) in $L^r(\tilde{B})\times R$. Thus we do not lose anything by working with $L^\infty(\tilde{B})$ rather than $L^r(\tilde{B})$.

Proof: *Step* 1. We prove the existence of $(u_n\ \lambda_n)$. First, we can shrink ε such that $\phi^2(\alpha)\neq 0$ for $|\alpha|\leqq\varepsilon$. We assume that $\alpha\phi'(\alpha)<0$ for $\alpha\neq 0$. The other case is similar. Choose $\alpha\in(-\varepsilon,0)$. Since $\phi'(\alpha)\neq 0$, the operator $-\Delta-\tilde{\phi}(\alpha)\,f'(\tilde{\psi}(\alpha))\,I$ is invertible as a map of T into $L^\infty(B)$ (cf. [14, Lemma 1]). Here $\tilde{\psi}(\alpha)=u_0+\alpha h_0+\psi(\alpha)$ and $\tilde{\phi}(\alpha)=\lambda_0+\phi(\alpha)$.

Hence, by Theorem 1, there exists a solution w_n of (11) (for $\lambda=\tilde{\phi}(\alpha)$ and $\Omega=\Omega_n$) close to $\tilde{\psi}(\alpha)$ in $L^r(\tilde{B})$ and the linearization at w_n is invertible. Consider the branch of solutions of (11) (for $\Omega=\Omega_n$) which starts at $(w_n,\tilde{\phi}(\alpha))$ as we increase λ. By Theorem 1 and the remarks at the start of this section, it must stay close to $\{(\tilde{\psi}(\alpha),\tilde{\phi}(\alpha))\colon\alpha\in(-\varepsilon,\varepsilon)\}$ in $L^r(\tilde{B})\times R$ until it gets close to $(\tilde{\psi}(\varepsilon),\tilde{\phi}(\varepsilon))$. In particularly by our assumptions on ϕ, the branch must change direction. Since solutions close to $\{(\tilde{\psi}(\alpha),\tilde{\phi}(\alpha))\colon \delta\leqq|\alpha|\leqq\varepsilon\}$ must have invertible linearizations (and this holds uniformly in α), we see that there is a solution (u_n,λ_n) of (11) for $\Omega=\Omega_n$ such that (u_n,λ_n) is close to (u_0,λ_0) in $L^r(\tilde{B})\times R$ and $-\Delta-\lambda_n f'(u_n)\,I$ is not invertible (for $\Omega=\Omega_n$). Since $-\Delta-\lambda_0 f'(u_0)\,I$ has a one-dimensional kernel, similar limit arguments to those in Step 3 of the proof of Theorem 1 show that $-\Delta-\lambda_n f'(u_n) I$ has at most a one-dimensional kernel for large n. (If there were a multidimensional kernel we could choose a basis which is L^2-orthogonal. In the limit, this would be an orthogonal family in the kernel of $-\Delta-\lambda_0 f'(u_0) I$). Note that, as earlier, the u_n are uniformly bounded in $L^\infty(\Omega_n)$ and so we do not have to worry about growth rates. Hence, for large n, the kernel of $-\Delta-\lambda_n f(u_n) I$ is spanned by h_n where $\|h_n\|_2=1$. As in the proof of Theorem 1, $h_n \to h_0$ in $L^2(\tilde{B})$ as $n\to\infty$. In addition, a similar argument to the proof of the boundedness of $\|u_n\|_\infty$ shows that $\|h_n\|_\infty$ is uniformly bounded. Hence $h_n \to h_0$ in $L^p(\tilde{B})$ for all $p<\infty$. Since $u_n \to u_0$ in $L^p(\tilde{B})$ for all $p<\infty$, it follows that $\int_{\Omega_n} f(u_n) h_n \to \int_B f(u_0) h_0$ as $n\to\infty$. (We can think of both integrals as integrals over $\tilde{B}$.) In particular, $\int_{\Omega_n} f(u_n) h_n \neq 0$ for large n and hence (u_n, λ_n) is a simple bifurcation point. This proves Step 1. Note that the same argument as in the last part of the proof shows that $\int_{\Omega_n} f''(u_n) h_n^3 \neq 0$ for large n and has the same sign $\int_B f''(u_0) h_0^3$.

Step 2: *Uniform existence of* ϕ_n *and* ψ_n. We look for solutions in the form $(u_n+\alpha h_n+w,\ \lambda_n+t)$ where $w\in R_n \equiv \{u\in T_n: \int_{\Omega_n} uh_n=0\}$. Here $T_n=\{u\in \dot{W}^{1,2}(\Omega_n): \Delta u\in L^\infty(\Omega_n)\}$. Define $P_n: T_n\to T_n$ by $P_n u=u-\langle u, h_n\rangle h_n$. Here $\langle\ ,\ \rangle$ is the scalar product on $L^2(\tilde{B})$. Let $A_n=-\Delta-\lambda_n f'(u_n) I$ (on T_n). Then $(u_n+\alpha h_n+w,\ \lambda_n+t)$ is a solution if and only if (w, t) is a solution of

$$\begin{aligned} w &= A_n^{-1} P_n[\lambda_n \operatorname{Re}_n(w+\alpha h_n)+tf(u_n+\alpha h_n+w)] \\ t &= (\langle f(u_n), h_n\rangle)^{-1}\,(-t\langle f'(u_n)(w+\alpha h_n), h_n\rangle \\ &\quad -(\lambda_n+t)\langle \operatorname{Re}_n(w+\alpha h_n), h_n\rangle) \end{aligned} \tag{12}$$

(cf. [1, p. 179]). Here when we write A_n^{-1} we mean the inverse of A_n as a map of R_n into $\tilde{R}_n=\{u\in L^\infty(\Omega_n): \langle u, h_n\rangle=0\}$ and $\operatorname{Re}_n(w)= f(u_n+w)-f(u_n)-f'(u_n)w$. It is easy to prove that there exist $\varepsilon_1, \varepsilon_2>0$ such that for each large n and for each $\alpha\in[-\varepsilon_1, \varepsilon_1]$ the right hand side of (12) defines a contraction mapping (with constant $\frac{1}{2}$) of $\{w\in\tilde{R}_n: \|w\|_\infty \leqq \varepsilon_2\}\times\{t\in R: |t|\leqq\varepsilon_2\}$ into itself if we prove that $\|A_n^{-1}\|$ is uniformly bounded where we consider A_n^{-1} as a map of $\tilde{R}_n$ into itself. Hence, if we prove this estimate, the contraction mapping principle will give the uniform existence of ψ_n and ϕ_n and their regularity follows as in the standard proof. (Note that f is a C^2 map of $L^\infty(\Omega_n)$ into itself.)

Thus we have to bound $\|A_n^{-1}\|$. We first reduce this to establishing the corresponding uniform bounds in L^2 (rather than L^∞). Assume the L^2 uniform bounds are true and there exist $Z_n\in\tilde{R}_n$ such that $\|Z_n\|_\infty=1$ and $\|y_n\|_\infty\to\infty$ as $n\to\infty$ where $y_n=A_n^{-1}Z_n$ (or at least this is true for a subsequence). Since $\|Z_n\|_2$ is uniformly bounded, the L^2 uniform bounds ensure that $\|y_n\|_2$ is uniformly bounded. By the definition of A_n,

$-\Delta y_n = f'(u_n)\, y_n + Z_n$. Hence $-\Delta y_n$ is uniformly bounded in $L^2(\Omega_n)$. By Lemma 1 we have a uniform bound on $\|y_n\|_p$ where $p = 2m(m-2)^{-1}$ ($p < \infty$ if $m = 2$). After a finite number of steps, we will obtain a uniform bound for $\|y_n\|_\infty$. Since this contradicts our assumption, it suffices to establish the uniform L^2 bound.

To prove the L^2 bound, note that, since A_n is self-adjoint, $\|A_n^{-1}\|^{-1}$ is $\inf\{|\lambda| : \lambda$ is a nonzero eigenvalue of $A_n\}$. This follows since A_n has a compact resolvent. Hence, if our results is false, there exist $v_n \in \dot{W}^{1,2}(\Omega_n)$ and $\mu_n \in R$ such that $\|v_n\|_2 = 1$; $A_n v_n = \mu_n v_n$, $\langle v_n, h_n \rangle = 0$ and $\inf\{|\mu_n|\} = 0$. By choosing a subsequence, we can ensure that $\mu_n \to 0$ as $n \to \infty$. By our assumptions, $\mu_n \neq 0$ for all n. Now $-\Delta v_n - f'(u_n)\, v_n = \mu_n v_n$. By passing to the limit as in the proof of Theorem 1, we obtain that a subsequence of v_n converges weakly in $\dot{W}^{1,2}(\tilde{B})$ to v_0 where $\|v_0\|_2 = 1$, $v_0 \in \dot{W}^{1,2}(B)$, $\Delta v_0 - f'(u_0)\, v_0 = 0$, and $\langle v_0, h_0 \rangle = 0$. (Remember that $\mu_n \to 0$ as $n \to \infty$, $h_n \to h_0$ in $L^2(\tilde{B})$ as $n \to \infty$, and $\langle v_n, h_n \rangle = 0$.) This contradicts our assumption that $-\Delta - f'(u_0)\, I$ has a one-dimensional kernel on B. Hence we have a contradiction and the result is proved. This completes Step 2 of the proof.

Completion of the proof. We need to prove that $\phi_n(\alpha) \to \phi(\alpha)$ and $\psi_n(\alpha) \to \psi(\alpha)$ in $L^r(\tilde{B})$ uniformly in α. Suppose that $\alpha_n \in [-\varepsilon_1, \varepsilon_1]$ for all n. We will prove that $(u_n + \alpha_n h_n + \psi_n(\alpha_n), \lambda_n + \phi_n(\alpha_n))$ converges in $L^r(\tilde{B}) \times R$. This will effectively complete the proof. To prove this, it suffices to show that every subsequence has a convergent subsequence and the limit is independent of the choice of subsequence. By our construction $|\phi_n(\alpha_n)| \leqq \varepsilon_1$ for all n and $\{\tilde{u}_n\}$ is uniformly bounded in $L^\infty(\tilde{B})$ where $\tilde{u}_n = u_n + \alpha_n h_n + \psi_n(\alpha_n)$. Thus, we can argue much as in the proof of Step 1 of the proof of Theorem 1. We find that by choosing a subsequence if necessary, we can ensure that $(\tilde{u}_n, \lambda_n + \phi_n(\alpha_n))$ converges in $L^r(\tilde{B}) \times R$ (and in fact in $L^p(\tilde{B}) \times R$ for all $p < \infty$) to a solution (v, β) of (11) for $\Omega = B$. Hence $\tilde{u}_n - u_n \to v - u_0$ in $L^r(\tilde{B})$ and thus, since $h_n \to h_0$ in $L^r(\tilde{B})$ and $\alpha_n \to \alpha$, $\psi_n(\alpha_n) \to v - u_0 - \alpha_0 h_0 \equiv w_0$ in $L^r(\tilde{B})$. Since $\langle \psi_n(\alpha_n), h_n \rangle = 0$, $\langle w_0, h_0 \rangle = 0$ while, since $\|\psi_n(\alpha_n)\|_\infty \leqslant \varepsilon_1$ for all n, $\|w_0\|_\infty \leqslant \varepsilon_2$. (Note that $\psi_n(\alpha_n)$ will converge to w_0 almost everywhere.) Hence $w_0 \in R_0$ and $\|w_0\|_\infty \leqq \varepsilon_2$. Moreover, since $|\phi_n(\alpha)| \leqq \varepsilon$, $|\beta - \lambda_0| \leqq \varepsilon_2$. Hence (v, β) is a solution such that $|\beta - \lambda_0| \leqq \varepsilon_2$ and $v = u_0 + \alpha h + w_0$ where $w_0 \in \tilde{R}_0$ and $\|w_0\|_\infty \leqq \varepsilon_2$. Hence by the uniqueness in the construction of $(\psi(\alpha), \phi(\alpha))$, $w_0 = \psi(\alpha)$, and $\beta = \lambda_0 + \phi(\alpha)$. Hence we have shown that a subsequence of $(\tilde{u}_n, \phi_n(\alpha_n))$ converges to $(u_0 + \alpha h_0 + \psi(\alpha), \phi(\alpha))$ in $L^r(\tilde{B}) \times R$ as $n \to \infty$. Hence the limit is independent of the choice of subsequence. Our earlier comments imply that $(\tilde{u}_n, \phi_n(\alpha_n)) \to (u_0 + \alpha h_0 + \psi(\alpha), \phi(\alpha))$ is $n \to \infty$. Hence, by the definition of $\tilde{u}_n$, $\psi_n(\alpha_n) \to \psi(\alpha)$ in $L^r(\tilde{B})$ and $\phi_n(\alpha_n) \to \phi(\alpha)$ as $n \to \infty$. Hence the uniform convergence follows. Note that it is possible to give a different proof of the last part by working directly with the equations and using that $A_n^{-1} \to A_0^{-1}$ in a suitable generalized sense (where A_0 is defined in the natural way).

Last, for this section, we consider one of the next simplest bifurcations. As a consequence, we will give examples where there is true secondary bifurcation (rather than a branch just changing direction) and show that this property is quite stable to perturbations provided that we restruct to domains with a suitable symmetry. Note that it will not be stable to domain changes where the symmetry is lost.

The equation we consider is

$$\begin{aligned} -\Delta u &= \lambda e^u && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{13}$$

where $\Omega \subseteq R^m$ with $m \leqq 8$. In fact, we can replace e^u by a large class of smooth superlinear convex nonlinearities $f(u)$. We assume that B is a union of *two* disjoint balls B_1, B_2 (possibly touching) of the same radius. B is invariant under the reflection $\tilde{H}$ in the hyperplane H with normal parallel to the line joining the centres p_1, p_2 of the two balls in B and such that the mid-point of p_1 and p_2 is in H. We now assume that the Ω_n are as before except we make the additional assumption that $\tilde{H}\Omega_n = \Omega_n$ for all n, that is, Ω_n is symmetric under the reflection symmetry $\tilde{H}$. For convenience, we assume that $p_1 + p_2 = 0$.

By results in Crandall and Rabinowitz [8] and Amann and Laetsch [2], there is a $\lambda_0 > 0$ such that (13) (for $\Omega = B$) has a positive solution $u_0(\lambda)$ for $0 \leqq \lambda \leqq \lambda_0$ such that (i) any other nonnegative solution u_1 satisfies $u_1 \geqq u_0(\lambda)$, (ii) $u_0(\lambda_0)$ is the unique nonnegative solution for $\lambda = \lambda_0$, (iii) there are no nonnegative solutions for $\lambda > \lambda_0$, and (iv) the linearization at $u_0(\lambda)$ is invertible for $\lambda < \lambda_0$. u_0 is known as the minimal solution. These results depend upon the convexity of f and that $m \leqq 8$. A word of explanation is required here. As in Section 1, to find the solutions on B, we merely have to find the solutions on each ball of B and add them together. (Note that our assumptions ensure that the balls in B have the same radius.) Note also that the results in [2, 8] assume that Ω is connected. Moreover, from the same ideas, we see that $-\Delta - \lambda_0 \exp u_0(\lambda_0)\, I$ has a two-dimensional kernel spanned by a function h_1 which is zero on B_2 and positive on B_1 and $h_2(x) = h_1(\tilde{H}x)$. Thus h_2 is zero on B_1 and positive on B_2.

Similar properties hold for Ω_n (which is connected). In this case $u_n(\lambda)$ denotes the minimal solution which is defined for $0 < \lambda \leqslant \lambda_n$. The one difference is that $-\Delta - \lambda_n \exp u_n(\lambda_n)\, I$ will have a one-dimensional kernel spanned by h_n where h_n is positive on Ω_n. The difference occurs because Ω_n is connected. Note also, by [2], $(u_n(\lambda_n), \lambda_n)$ is the only solution of (13) for

$\Omega=\Omega_n$ for which the linearization is degenerate with a nonnegative function in the kernel. Last, note that u_n (and also u_0) are even in the sense that $u_n(\lambda)(\tilde{H}x)=u_n(\lambda)(x)$ for $x\in\Omega_n$ and $0\leqq\lambda\leqq\lambda_n$. This follows because otherwise $u_n(\lambda)(\tilde{H}x)$ would also be a solution of (13) which is not larger than the minimal solution.

We now consider our problem in the space of even functions. Thus we consider solutions in $L^{\infty,e}(\Omega_n)=\{u\in L^\infty(\Omega_n): u \text{ is even}\}$. In this space, on B, (13) has solutions $\{u_0(\lambda): 0\leqq\lambda\leqq\lambda_0\}$ such that the linearization at $u_0(\lambda)$ is invertible for $0\leqq\lambda<\lambda_0$. Moreover, $(u_0(\lambda_0), \lambda_0)$ is a simple bifurcation point in our earlier sense *if we work in the space of even functions.* Technically, we consider our map as a map of $T^e\times R\to L^{\infty,e}(B)$ where T^e denotes the even functions in T. That $(u_0(\lambda_0), \lambda_0)$ is a simple bifurcation point follows, because at $(u_0(\lambda_0), \lambda_0)$, the kernel in the space of even functions is spanned by $\tilde{h}=\frac{1}{2}(h_1+h_2)$. Since this is positive on B, one easily sees that $\int_B \tilde{h}\exp u_0>0$ and the simplicity of the bifurcation point follows. Note that the standard theory [2] applied to B_1 shows that the branch of even solutions "bends back" at (u_0, λ_0). In fact $\phi''(0)<0$ where ϕ is defined as earlier. Thus by our earlier perturbation results applied to $\{(u_0(\lambda), \lambda): 0\leqq\lambda\leqq\lambda_0\}$ *in the space of even functions* we see that, for large n, the branch of minimal solutions of (13) for $\Omega=\Omega_n$ exists up to λ_n where λ_n is close to λ_0 and then bends back. Note that it is easy to generalize our earlier results to the space of even functions. In particular, we see that $\lambda_n\to\lambda_0$ as $n\to\infty$, $\{u_n(\lambda): 0\leqq\lambda\leqq\lambda_n\}$ are uniformly bounded in $L^\infty(\tilde{B})$ and $u_n(\lambda_n)\to u_0(\lambda_0)$ in some exponential Orlicz space as $n\to\infty$. Moreover, by our results on simple bifurcation, there is an $\varepsilon>0$ independent of n such that the even solutions close to (u_n, λ_n) in $L^\infty(\Omega_n)\times R$ are $\{(u_n+\alpha h_n+\psi_n(\alpha), \lambda_n+\phi_n(\alpha)): |\alpha|\leqq\varepsilon\}$. Here $\psi_n(0) = 0$ and $\phi_n(0) = 0$. Here we use our earlier notation. Moreover, these solutions are nondegenerate in the space of even functions for $\alpha\neq 0$. To see this, note that the theory of convex operators [2] implies that for $\alpha\neq 0$, a degenerate solution must have a corresponding element of the kernel which changes sign in Ω_n (and which is even). Hence, if this occurred for a sequence of large n's, we can find a sequence $\alpha_n\in[-\varepsilon,\varepsilon]$ such that $\alpha_n\to\alpha$ as $n\to\infty$ and some higher eigenvalue of $-\Delta-\tilde{\phi}_n(\alpha_n)\exp(\tilde{u}_n)I$ on $L^{2,e}(\Omega_n)$ is zero. Here $\tilde{u}_n=u_n+\alpha_n h_n+\psi_n(\alpha_n)$ and $\tilde{\phi}_n(\alpha)=\lambda_n+\phi_n(\alpha)$. Since $\tilde{u}_n\to u(\alpha)\equiv u_0+\alpha h_0+\psi(\alpha)$ in a suitable Orlicz space and $\phi_n(\alpha_n)\to\phi(\alpha)$ by Theorem 3, we can argue as in the proof of Step 3 of Theorem 1 to deduce that the eigenvalues of $Z_n(\alpha_n)=-\Delta-\tilde{\phi}_n(\alpha_n)\exp(\tilde{u}_n)I$ on $L^{2,e}(\Omega_n)$ approach those of $Z(\alpha)=-\Delta-\tilde{\phi}(\alpha)\exp(u_0+\alpha h+\psi(\alpha))I$ on $L^{2,e}(B)$ counting multiplicity. Now the last operator is invertible for small nonzero α (cf. [2] or [7]) and hence our assumption of a zero eigenvalue implies that $\alpha=0$. Our assumption implies that the second eigenvalue v_n of $Z_n(\alpha_n)$ is nonpositive. Hence the second eigenvalue v of $Z(\alpha)$ is nonpositive (by continuous dependence). This is impossible because, if $\alpha=0$, the first eigenvalue of $Z(\alpha)$ is zero and simple. (Remember that we are working in a space of even functions.) Hence our invertibility claim is proved.

We now consider our operators on all of L^2 (rather than on even functions). Consider $Z(\varepsilon)$. One easily sees that $Z(\varepsilon)$ has exactly two negative eigenvalues if ε is small. Note that $Z(\varepsilon)$ splits up as a direct sum of two operators. Thus, by continuous dependence $Z_n(\varepsilon)$ has two negative eigenvalues and no zero eigenvalue for n large. (This contrasts with $Z_n(0)$ which has a simple zero eigenvalue and no negative eigenvalues.)

We can now prove that there is a secondary bifurcation point between $\tilde{u}_n(\varepsilon) = u_n + \varepsilon h_n + \psi_n(\varepsilon)$ and u_n on the branch of even solutions. To prove that, we will show that

$$\operatorname{index}((\lambda_n + \phi_n(\varepsilon))\, A_n, \tilde{u}_n(\varepsilon)) = 1, \tag{14}$$

where $A_n(z) = (-\Delta)^{-1} \exp z$. Here A is considered as a map of $L^\infty(\Omega_n)$ into itself. One easily shows that A_n is compact since the natural map of $L^\infty(\Omega_n) \to W^{-1,p}(\Omega_n)$ is compact. Assuming (14) for the moment, we will prove secondary bifurcation. Now, as is well known [1] the minimal solution has index 1 for $\lambda < \lambda_n$. Since our problem has exactly 2 solutions close to u_n for λ near λ_n but less than λ_n, the additivity of the degree ensures that $\operatorname{index}((\lambda_n + \phi_n(\alpha))\, A_n, \tilde{u}_n(\alpha)) = -1$ for small positive α. Hence index $(\tilde{\phi}_n(\alpha)\, A_n, \tilde{u}_n(\alpha))$ changes sign between small positive α and $\alpha = \varepsilon$ and thus there must be secondary bifurcation off $\{(\tilde{u}_n(\alpha), \tilde{\phi}_n(\alpha)) : 0 \leqq \alpha \leqq \varepsilon)\}$. As n gets large this must occur arbitrarily close to λ_n (since we can choose ε small and since ϕ_n converges uniformly to ϕ).

It remains to prove (14). By a standard formula for the index [30], it suffices to prove that $(\lambda_n + \phi_n(\varepsilon))\, A_n'(\tilde{u}_n(\varepsilon))$ has exactly two eigenvalues larger than 1 (counting multiplicity) and that 1 is not an eigenvalue. Now the eigenvalues of this operator are the reciprocals of the eigenvalues of the problem $-\Delta h = \lambda(\lambda_n + \phi_n(\varepsilon)) \exp(\tilde{u}_n(\varepsilon))\, h$ in $\dot{W}^{1,2}(\Omega_n)$ counting multiplicity. (Here we can easily check that the change of space does not matter.) Thus we need to prove that the problem

$$-\Delta h - (\lambda_n + \phi_n(\varepsilon)) \exp(\tilde{u}_n(\varepsilon))\, h = \lambda(\lambda_n + \phi_n(\varepsilon)) \exp(\tilde{u}_n(\varepsilon))\, h$$

has exactly two negative eigenvalues. This follows from our earlier results on $Z_n(\varepsilon)$ once we note that, if L is self-adjoint with compact resolvent and invertible and if B is positive, bounded and self-adjoint, then the number of negative eigenvalues of $Lx = \lambda Bx$ is independent of B. This follows easily from the variational characterization of eigenvalues.

We have proved the following theorem.

THEOREM 4. *Assume that our basic assumptions on the Ω_n hold, that our symmetry assumption on the Ω_n holds, and that $m \leqq 8$. Then, for n large, there is a second bifurcation off the branch of even solutions of* (13) *and this secondary bifurcation occurs close to λ_n if n is large.*

Remarks. 1. With more care, one can make a complete local analysis of the solutions close to (u_n, λ_n). It turns out that nearby there is exactly one point of secondary bifurcation and there is a true pitchfork bifurcation with branching to the left and no further "bending" of the solution curves. Here, what is meant by close is uniform in n (See Fig. 1).

The idea to prove this is to study the two-dimensional bifurcation equation for solutions close to (u_n, λ_n). One estimates its leading coefficients by using that they are close to the corresponding coefficients on B (where they are easier to calculate). One also uses the Z_2 symmetry and uses the complex degree to show that there can be at most four solutions (locally) for each nearby λ.

2. Note that we only need the convexity of f for $0 \leqq y \leqq \|u_0\|_\infty$ and that one could prove secondary bifurcation without using the convexity at all (though, we would not have as detailed a local understanding).

3. Our method could be used if B consists of k-balls of equal radius and Ω_n is invariant under a suitable rotation. However, this time one needs to use a homotopy index argument to prove that there is secondary bifurcation. One can get more information by working in suitable invariant subspaces formed using the symmetries but the detailed local solution structure is unclear.

4. Note that, by the results of Saut and Teman [33], secondary bifurcation is not generic.

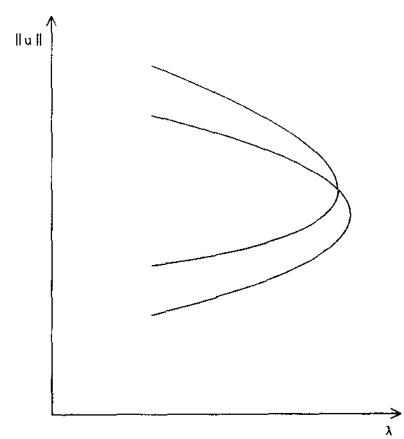

FIGURE 1

3. Examples

In this section, we apply the results of Sections 1 and 2 to a number of well-known equations and show that the number of positive solutions depends on the domain shape. In fact, our results seem to essentially say that the only case where the number of positive solutions can possibly be independent of domain shape is when the nontrivial positive solution on the ball is unique and the trivial solution (if it exists) is an unstable solution of the natural corresponding parabolic on the ball.

We first consider the equation

$$\begin{aligned} -\Delta u = u^p \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega, \end{aligned} \tag{15}$$

where $1 < p < (m+2)(m-2)^{-1}$ ($1 < p < \infty$ if $m=2$). First, assume that Ω is a ball. Then, by combining results in [10, 17], we see that (15) has a unique positive solution on a ball and this solution is nondegenerate. It follows that our problem has exactly $2^s - 1$ nontrivial positive solutions on B and these are all nondegenerate. Here s is the number of disjoint balls which form B. Note that we can choose the solution zero on some but not all of the balls. Hence, since y^p has zero derivative at zero, Theorem 2 implies that, on Ω_n for n large, (15) has a large number of positive solutions with invertible linearization. In fact, there are exactly $2^s - 1$ nontrivial positive solutions with norm bounded in $L^r(\tilde{B})$ and each of these is nondegenerate. We will discuss large solutions in Section 5. In the above examples, Ω_n can be chosen contractible. By choosing B to be two touching balls, we can find examples where Ω_n is star-shaped and there are three nontrivial nonnegative solutions (and which are nondegenerate). Thus the star-shapedness of Ω is not enough to ensure uniqueness. This shows that the conditions to ensure uniqueness for $p < (m+2)(m-2)^{-1}$ are different from the conditions which ensure nonexistence of a nontrivial positive solution for $p = (m+2)(m-2)^{-1}$.

Second, we consider the Gelfand equation

$$\begin{aligned} -\Delta u = \lambda e^u \qquad & \text{in } \Omega \\ u = 0 \qquad & \text{on } \partial\Omega \end{aligned} \tag{16}$$

for $\Omega \subseteq R^m$ with $m \leqslant 8$. We assume that the Ω_n satisfy the assumptions at the beginning of Section 1. First, assume that $m=2$. Now on a ball it is known that (16) has exactly two solutions for $\lambda < \lambda_0$ and these two solutions are nondegenerate. Moreover, as earlier, there is no solutions for $\lambda > \lambda_0$ and a unique solution for $\lambda = \lambda_0$ (and this last point is a simple bifurcation point). Moreover the solution structure looks as in Fig. 2.

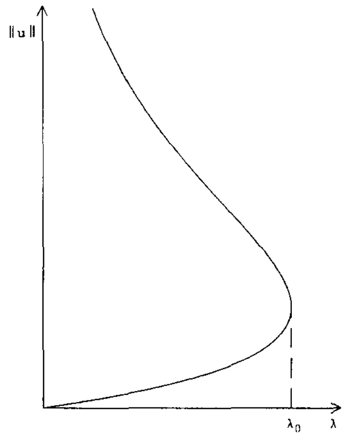

FIGURE 2

It follows easily from Theorem 2 that for all but a finite number of λ's in $(0, \lambda_0)$, there will be 2^s solutions of (16) for $\Omega = \Omega_n$ and n large and these are the only solutions not of large L^∞ norm. Here λ_0 was defined in Section 2. The finite number of λ's are near to the λ_0's for the various balls making up B. Moreover, we can obtain more precise results. First, assume that $s = 2$ and that the two balls making up B are of different radius. On B, it is easy to see that (15) has exactly two simple bifurcation points and all other points are nondegenerate points. (Remember that on B our system is really a "direct sum" of the system on the two balls.) Hence we can apply our theory in Section 2. We find that for $\lambda \geqslant \alpha$ for some fixed $\alpha > 0$, the solutions for $\Omega = \Omega_n$ not of large norm in $L^\infty(\tilde{B})$ look as in Fig. 3.

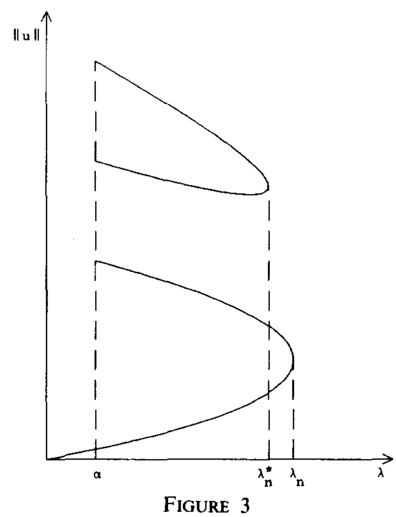

FIGURE 3

Here λ_n and λ_n^* are close to λ_0 and $\lambda_n^* < \lambda_n$. The above result follows because a similar result holds on B. Note that the above diagram is slightly inaccurate in that it is not clear that the smaller of the two solutions on the top branch has larger norm than the larger of the two solutions on the bottom branch. If the two balls have nearly equal radius, the point (u_n^*, λ_n^*) where the top branch turns around is close to the bottom branch. If the balls have very different radii, then the two branches are not close, and all the solutions on the top branch have large norm. This follows because all the solutions on the small ball for fixed λ will have large norm. (This last result follows because these solutions can be obtained by rescaling the ones on the ball of radius 1.) It is an interesting open question to decide whether the two solution branches join up at small λ. (As n gets large, the point where they join up must get smaller and smaller if it occurs at all.) A similar result holds if B consists of more than two balls provided they each have different radii. (Of course, there are now more solutions.)

Now consider the case where $s = 2$, the balls have equal radius, and each of the Ω_n's have the same reflection symmetry as at the end of Section 2. In this case, if we fixed $\alpha > 0$ and look at the solution for $\lambda \geqslant \alpha$ and not of large norm, the results in Section 2 imply that our solutions structure will be as in Fig. 4, where the dotted solutions do not preserve the symmetry while the others do.

Note that λ_n^* and λ_n are both close to λ_0. This shows that, for the Gelfand equation on star-shaped domains, secondary bifurcation can occur arbitrarily close to the first turning point. It would be interesting to know if the nonsymmetric branches join up with the symmetric branch for small λ.

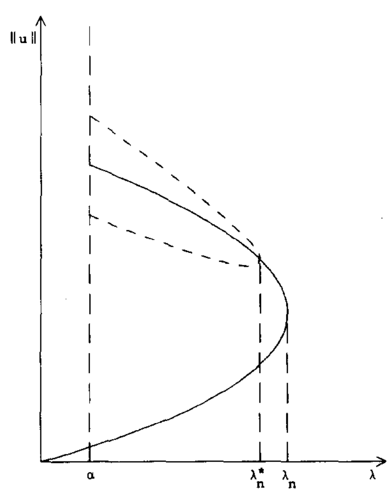

FIGURE 4

One might ask what happens if the balls have equal radii but the Ω_n's do not preserve the symmetry.

Now consider briefly the case where $3 \leqslant m \leqslant 8$. Here one can prove the existence of many solutions and of secondary bifurcation under similar hypotheses as before. However, the solution structure is much more complicated because the solution structure on a ball is much more complicated. Recall that, on a ball, the solution structure is as in Fig. 5.

Moreover, by results in [9, 12], all points are nondegenerate points or simple bifurcation points. Let us assume B consists of two balls and each Ω_n is star-shaped. If Ω_n has the H-symmetry, a modification of our earlier ideas implies that there are arbitrarily many points of secondary bifurcation while, if the two balls in B have different radii, there exist two distinct λ's where there are many solutions (where the number increases with n). It would be interesting to know whether, in this case, there are actually an infinite number of solutions for these two λ's. It is clearly possible to obtain more detailed information on the solution structure in the above cases.

Before we leave our first two examples, note that in these two cases and if the domain is symmetric one can give a short proof of the existence of multiple solutions by considering a constrained maximization problem and showing that the maximum occurs at a point which breaks the symmetries. Similar arguments appear in [11, 34]. However, the present methods work more generally and give more information.

Third, we consider the problem

$$\begin{aligned} -\Delta u &= \lambda f(u) \qquad \text{in } \Omega \\ u &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{17}$$

where $f(0)=0$, $f'(0)=1$ and where $y^{-1}f(y)$ is strictly decreasing on $[0, \infty)$, e.g., $f(y)=y-y^3$. Then it is well known that for $\lambda > \lambda_1(\Omega)$ (17) has a unique nontrivial positive solution $x_1(\lambda, \Omega)$. Here $\lambda_1(\Omega)$ denotes the

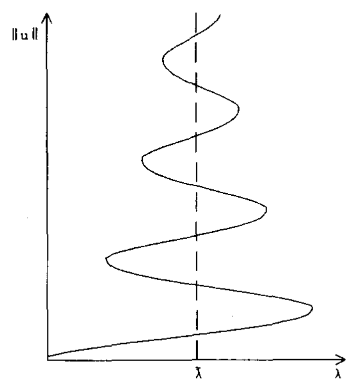

FIGURE 5

smallest eigenvalue of $-\Delta$ on Ω. At first glance, our earlier construction seems to produce at least three nontrivial positive solutions on one of our domains Ω_n. However, if we take the solution on B which is $x_1(\lambda, B_1)$ on one of the balls B_1 making up B and is zero on the other (where for simplicity, we are assuming $s=2$), we see from Theorem 2 that, if $\lambda > \lambda_1(B_2)$, then the solution $\tilde{x}(\lambda)$ near it must change sign on Ω_n. Here we are assuming that $\lambda > \lambda_1(B_1)$. By using Remark 3 after Theorem 1, we see that $\tilde{x}(\lambda)$ will be near $x_1(\lambda, B_1)$ on most of B_1 and will be small on most of B_2 and sometimes negative. This type of solution does not seem to have been observed before. Under additional assumptions on the Ω_n, we could say more about these solutions by using the ideas in Remark 3 after Theorem 1.

Fourth, we consider (17) again but now assume that $f(0)=0, f'(0)=1$, $0<a_1<a_2<a_3$, $f(y)>0$ on $(0, a_1)$, f vanishes at a_1, a_2, a_3, $f(y)<0$ on (a_1, a_2), $f(y)>0$ on (a_2, a_3), and $\int_{a_1}^{a_3} f(s)\, ds>0$. For simplicity assume that $s=2$. If λ is sufficiently large ($\lambda > \tilde{\lambda}_1$), it is known [36] that (17) (for $\Omega = B_1$) has a solution $u_1(\lambda)$ such that the solution is nondegenerate and stable and is the unique positive solution with L^∞ norm close to a_3 but less than a_3. Similarly by [13], if $\lambda > \tilde{\lambda}_2$, there is a unique nontrivial solution $u_2(\lambda)$ on B_2 such that $0 \leqslant u_2(\lambda) \leqslant a_1$. Moreover, this solution is nondegenerate and stable. Fix $\lambda > \max\{\tilde{\lambda}_1, \tilde{\lambda}_2\}$. If n is large, Theorem 2 implies that there is a unique positive solution close to the solution of (17) which is near $u_1(\lambda)$ on B_1 and $u_2(\lambda)$ on B_2. Moreover this solution is nondegenerate and *stable*. In particular, we have examples where Ω_n has a Z_2 symmetry and is star-shaped but there are *stable solutions which do not have the* symmetry *and persist for relatively large* λ (in particular, λ's much larger than $\lambda_1(\Omega_n)$). This holds nonuniformly in n because for fixed n, the result in [34] implies the uniqueness of the solution which has L^∞ norm close to a_3. This example suggests that the *uniqueness* of the solution with L^∞ norm close to a_3 occurs only for much larger λ than for the nontrivial solution in the order interval $[0, a_1]$. (Note that for λ as above, our theory shows that there is a unique nontrivial nonnegative solution of (17) for n large in the order interval $[0, a_1]$). In fact, it can be shown that there are at least nine nontrivial solutions in the order interval $[0, a_3]$ if λ is as above and n is large. The above examples are also of interest in that they show the limits of the Gidas–Ni–Nirenberg Theorem [19].

4. Further Examples

In this section, we continue to apply our ideas to examples. In this section we apply our ideas to three equations we have studied in early papers and which appear in applications.

First, we consider the positive solutions of

$$\begin{aligned} -\Delta u &= \lambda f(u) && \text{in } \Omega, \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{18}$$

for large λ. Here $f(0)=0$, f is C^1, and there is a $p>1$ such that $y^{-p}f(y) \to a > 0$ as $y \to 0$. In addition, we assume that $p < (m+2)(m-2)^{-1}$ if $m > 2$. If $\Omega = \Omega_n$ for n large enough we prove that (18) has multiple small positive solutions for all large λ. Here Ω_n satisfies the conditions of Section 1. To see this, we look for solutions of the form $(a\lambda)^{-1/(p-1)} v$. Then v must satisfy.

$$\begin{aligned} -\Delta v &= v^p + v^p r((\lambda a)^{-1/(p-1)} v) && \text{in } \Omega \\ v &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{19}$$

where $f(y) = ay^p(1+r(y))$. Suppose that v_0 is a nontrivial nonnegative solution of

$$\begin{aligned} -\Delta v &= v^p && \text{in } \Omega, \\ v &= 0 && \text{on } \partial\Omega \end{aligned} \tag{20}$$

which is nondegenerate. Then, for all sufficiently large λ, there is a positive solution of (19) near v_0. To see this, note that v_0 is positive in Ω by the maximum principle and thus is easily seen to be demi-interior to $L^q(\Omega)$ for all q (where demi-interior is defined in [14]). If q is large, the map $A(v) = (-\Delta)^{-1} v^p$ is a completely continuous map of $L^q(\Omega)$ into $L^\infty(\Omega)$, by standard regularity theory, as earlier. Hence, we see that v_0 has the same index when considered as a fixed point of A in $L^\infty(\Omega) \cap K$ or in $L^q(\Omega) \cap K$, where K denotes the set of nonnegative functions in $L^1(\Omega)$. Now A is Fréchet differentiable on $L^q(\Omega)$ by using Vainberg [37, footnote on p. 168]. Since $I - A'(v_0)$ is invertible by our assumptions and since v_0 is demi-interior to $L^q(\Omega)$, Theorem 1 in [14] ensures that v_0 has nonzero index in $L^q(\Omega) \cap K$ and thus $L^\infty(\Omega) \cap K$. Since $y^p r(\lambda a)^{-1(p-1)} y) \to 0$ as $\lambda \to \infty$ uniformly on bounded sets (and thus the extra term in (19)) is a perturbation which is small in $L^\infty(\Omega)$), an elementary degree argument shows that there is a solution of (19) in $L^\infty(\Omega) \cap K$ close to v_0 for all large λ. Thus, we see that, for large λ, the number of nontrivial positive small solutions of (18) is at least as large as the number of nontrivial positive solutions of (20) which are nondegenerate. Hence our claim follows from the discussion at the beginning of Section 3.

By our earlier remarks we see that there may be at least three small positive solutions for a star-shaped Ω. If $y^{1-p}f'(y) \to pa$ as $y \to 0$, it is not difficult to show that there is a *unique* solution of (18) near v_0 for large λ.

In some cases, one can combine this result with the results in [13] to determine the exact number of positive solutions of (18) for large λ.

In addition, to our assumptions on f, assume that $f(y) \to C > 0$ as $y \to \infty$ and $yf'(y) \to 0$ as $y \to \infty$. For Ω a ball, the results in [13] imply that for large λ there is a positive solution $u_1(\lambda)$ of large norm and which is non-degenerate. Hence, by Theorem 2, if $\Omega = \Omega_n$ for n large and $s = 2$, there is a *stable* positive solution which is close to $u_1(\lambda)$ on B_1 and close to zero on B_2 (where $B = B_1 \cup B_2$). This shows that, in the results in [13] on the number of solutions for large λ, what is meant by large λ depends essentially on the shape of the domain (even for star-shaped domains). If $y^{1-p}f'(y) \to ap$ as $y \to 0$, we can use Theorem 2 and some simple a priori bounds to find exactly how many solutions (2) has for $\Omega = \Omega_n$ with n large (and fixed but large λ). Once again, there seems to be an interesting two-parameter problem here.

Second, we consider the nonnegative solutions of system

$$\begin{aligned} -\Delta u &= u(a - bu - cv) && \text{in } \Omega \\ -d\,\Delta v &= v(e - fu - gv) && \text{in } \Omega \\ u &= v = 0 && \text{on } \partial\Omega. \end{aligned} \tag{21}$$

This system is studied in [10] and arises in competing species models with diffusion. Here a, b, c, d, e, f, g are positive constants. In fact, we rescale so that $b = g = 1$ but we will not bother to do this (since it was not done in [10]). We take the other constants fixed and let d tend to zero. Then it is proved in [10] that, for small positive d, the number of strictly positive solutions of (21) is at least as large as the number of nontrivial positive isolated solutions of nonzero index (in the natural cone K) for the equation

$$\begin{aligned} -\Delta u &= uk(u) && \text{in } \Omega \\ u &= 0 && \text{on } \partial\Omega. \end{aligned} \tag{22}$$

Here $k\colon R \to R$ is defined by $k(y) = a - by - cg^{-1}(e - fy)^+$. Note that a solution of the system (21) is said to be strictly positive if both components are positive in all of Ω. Suppose $a = cg^{-1}e$. Then $yk(y) = (cg^{-1}f - b)\,y^2$ for $y \leqslant f^{-1}e$. Thus by a suitable choice of coefficients, we can ensure that $yk(y) = y^2$ for $y \leqslant \tilde{K}$ where $\tilde{K}$ is large. Thus, by the results of Section 3, we can construct Ω such that (22) has many nondegenerate positive solutions *if* $m \leqslant 5$. Thus, as before, these solutions have nonzero index in K and hence, for this choice of coefficients, (21) has many strictly positive solutions for all small positive d. Once again we need that $m \leqslant 5$.

There is a second method for constructing many strictly positive solutions by working directly with (21). We sketch this. Suppose

$(u_0, v_0) \in K \oplus K$ is a solution of (21) of nonzero index in $K \oplus K$ for $\Omega = B_1$ (in the space $L^\infty(B_1) \oplus L^\infty(B_1)$) and that (u_1, v_1) is a similar solution for $\Omega = B_2$. Let $B = B_1 \cup B_2$. Then, much as before, one can show that there is a nonnegative solution (u_n, v_n) of (21) for $\Omega = \Omega_n$ and n large near $(\tilde{u}, \tilde{v})$ where

$$\tilde{u}(x) = \begin{cases} u_0(x) & x \in B_1 \\ u_1(x) & x \in B_2 \end{cases}$$

and $\tilde{v}$ is defined analogously. Here Ω_n are as in Section 1. Moreover, one easily sees that (u_n, v_n) is a strictly positive solution if (i) u_0 or u_1 does not vanish identically *and* (ii) v_0 or v_1 does not vanish identically. Using this result, it is easy to give examples showing that the number of strictly positive solutions (and even the number of stable strictly positive solutions) can be large whenever the solutions $(\bar{u}, 0)$ and $(0, \bar{v})$ both exist (and neither is $(0, 0)$), are both nondegenerate, *and at least one of them* is stable (for the natural corresponding parabolic). Note that, as in [10], the stability of $(\bar{u}, 0)$ determines whether it has nonzero index in $K \oplus K$ and that, in the most interesting case where $(\bar{u}, 0)$ and $(0, \bar{v})$ are both unstable, our method does *not* imply the existence of many solutions. Note that the arguments of this paragraph do not need a restriction on m.

Third, consider the equations.

$$\begin{aligned} -\Delta u &= \alpha(1+v)^p \exp(-\gamma(1+u)^{-1}) \\ \Delta v &= -\alpha\beta(1+v)^p \exp(-\gamma(1+u)^{-1}) \\ \frac{\partial u}{\partial n} + \mu u = 0, &\qquad \frac{\partial v}{\partial n} + \nu v = 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{23}$$

where $\alpha, \beta, \gamma > 0$, $p \geqslant 1$, $\mu, \nu \geqslant 0$, and where μ, ν may be $+\infty$ (that is, Dirichlet boundary conditions). These equations occur in chemical catalysis theory (cf. Aris [4]). In his notation, we have replaced u by $1+u$ and v by $1+v$. We will mainly consider the case where $\mu = \nu = \infty$ though we will make some remarks on generalizations at the end. Henceforth assume that $\mu = \nu = \infty$ unless explicit mention is made to the contrary. Then $\beta u + v$ is harmonic in Ω and vanishes on $\partial\Omega$. Thus $\beta u + v = 0$ on Ω. By eliminating v from the first equation and using the change of variables $u = \gamma^{-1} w$, $\alpha = \lambda\gamma^{-1} \exp \gamma$, we find that

$$\begin{aligned} -\Delta w &= \lambda f_\gamma(w) \qquad \text{on } \Omega \\ w &= 0 \qquad \text{on } \partial\Omega, \end{aligned} \tag{24}$$

where $f_\gamma(y) = (1 - \gamma^{-1}\beta y)^p \exp(y(1+\gamma^{-1}y)^{-1})$. In the applications, it is natural to look for solutions such that $0 \leqslant u(x) \leqslant \gamma\beta^{-1}$ in Ω. We can always ensure this by modifying f for $y \geqslant \gamma\beta^{-1}$. Note that $f_\gamma(y) \to \exp y$ as $\gamma \to \infty$ uniformly on bounded sets (in y). By Section 3, we can choose Ω such that the Gelfand equation has a large number of nondegenerate solutions for some fixed λ. Since $f_\gamma(y) \to \exp y$ as $\gamma \to \infty$ uniformly on bounded sets, a simple degree theory argument shows that there is a solution of (24) near each of these solutions of the Gelfand equation for γ large. In particular, for this Ω, one sees that if γ is large (and λ, β are fixed), one may have many solutions of (23). This contrasts with the case where Ω is a ball if $m=2$. Since $f_\gamma(y)$ is convex for y bounded if γ is large, one can use Remark 2 after Theorem 4 to show that, for some star-shaped domains, the set of solutions of (23) may contain secondary bifurcations when we fix γ, β and use λ (or α) as a parameter.

We can also use our ideas in a different way for this equation. Assume that $m=2$. For Ω a ball and a suitable large γ and suitable λ not small, it is shown in [9] that (23) has exactly three solutions, exactly two of which are stable and all are nondegenerate in the space of radially symmetric functions. Here, in the notation of [4], we are assuming that $L=1$ in the time-dependent equations. However, since $f_\gamma(u(x)) \geqslant 0$ on B_1 for all our solutions, the results in [12] imply that the linearization at a solution cannot have a nonsymmetric function in its kernel. Thus the three solutions are nondegenerate in $\dot{W}^{1,2}(B)$. Hence for Ω_n as in Section 2 with all the B_i's the same radius as B_1 and λ, β, γ as above, we find that there are exactly 3^s solutions and exactly 2^s are stable. Thus, in particular, we have examples where Ω_n is star-shaped and there are four stable solutions. Two of these solutions (the ones other than the maximal and minimal solutions) need not have the same symmetries as the domain. In most cases where $s=2$ one can use our ideas to obtain a complete picture of the solution structure for $\varepsilon \leqslant \lambda \leqslant \kappa$ if γ and n are large. (The difficult case is where B_1 and B_2 have the same radius but the Ω_n do not satisfy the conditions of Section 2.)

The equations (23) are the equations for a single pellet of catalysis. There will usually be many pellets of catalysis of varying shape. Our results indicate that the solution structure and even the number of stable solutions depends on the pellet shape. Thus, it seems to me that one will have to consider more carefully the meaning of this model.

Last, one might ask how important are the assumptions that $\mu = \nu = \infty$. First, one can use a perturbation argument to give examples of multiple stable solutions with μ and ν both large but finite (and not necessarily equal). Second, our perturbation off the Gelfand equation is valid if $\mu = \infty$ and ν is finite (or even if μ is sufficiently large). To do this, note that we can always reduce (23) to the equivalent single equation

$$-\Delta u = \alpha(1 - \beta u + P(u))^p \exp(-\gamma(1+u)^{-1}) \qquad \text{in } \Omega$$

$$\frac{\partial u}{\partial n} + \mu u = 0.$$

Here $S = P(u)$ is the unique harmonic function on Ω which satisfies the boundary condition $\partial S/\partial n + \nu S = \beta(\nu - \mu)\, u$. (If $\nu = \infty$, the boundary condition becomes $S = \beta u$ on $\partial\Omega$ while, if $\mu = \infty$, the boundary condition becomes $\partial S/\partial n + \nu S = \beta(\partial u/\partial n)$ on $\partial\Omega$). If we use the change of variable $u = \gamma^{-1} w$, $\alpha = \lambda\gamma^{-1} \exp\gamma$, our equation becomes

$$-\Delta w = \lambda(1 - \beta\gamma^{-1} w + \gamma^{-1} Pw)^p \exp(w(1 + \gamma^{-1} w)) \qquad \text{in } \Omega$$

$$\frac{\partial w}{\partial n} + \mu w = 0.$$

Once again we can think of this as a small perturbation of the Gelfand equation and hence for γ large it behaves like the Gelfand equation with multiple solutions and with secondary bifurcations if Ω_n has the symmetries of Section 2 (and Ω_n is reasonably smooth if μ is finite). One has to be a little more careful if $\nu = \infty$ and μ is finite in the choice of spaces. (Remember that we are assuming that $\mu = \infty$ or μ is large.)

5. Some Remarks on Large Positive Solutions

Suppose that Ω_n are as in Section 1. We are interested in when the solutions obtained in Theorem 2 give all positive solutions of (1) for $\Omega = \Omega_n$, under the assumption that there are only a finite number of positive solutions, all nondegenerate for $\Omega = B$. By the results of Section 1, we see that the only way this can fail is that there exist positive solutions u_n of (1) for $\Omega = \Omega_n$ such that $\{\|u_n\|\}_{n=1}^{\infty}$ is not bounded. Thus our problem reduces to the establishment of an a priori bound in L^∞ for the positive solutions of (1) which holds uniformly in n for $\Omega = \Omega_n$.

If f is sublinear, this a priori bound is very easy to establish (because Lemma 1 implies that we have a bound for the norm of $(-\Delta)^{-1}$ as a map of $L^\infty(\Omega_n)$ into itself which is independent of n). If $f(y) \leqslant 0$ for large positive y, the weak maximum principle easily implies a bound for $\|u\|_\infty$ which is independent of n. If f is asymptotically linear, it is also usually easy to obtain bounds which are uniform in n.

If f is superlinear, the problem is quite different and the result depends very much on the geometry of Ω_n. We consider the case $f(y) = y^p$ with $1 < p < (m+2)/(m-2)$ though with care our arguments work for rather more general nonlinearities.

Suppose we have Ω_n smooth satisfying the assumptions of Theorem 1 for which the positive solutions are uniformly bounded. Now by a simple scaling argument the unique solution u_R of (1) (for $f(y) = y^p$) on the ball B_R has the property that $\|u_R\|_\infty \to \infty$ as $R \to 0$. Choose R_n such that $R_n \to 0$ as $n \to \infty$ and $\|u_{R_n}\|_\infty \geqslant n$. Define Ω_n' to be $\Omega_n \cup (B_{R_n} + x)$, where x is chosen such that the two parts intersect and such that x is close to x_0 where $\bar{\Omega}_n$ intersects $\bar{B}_{R_n} + x_0$ at a single point. (Since $\partial\Omega_n$ is smooth and R_n is small, we can easily achieve this. If necessary, we can shrink R_n.) By the argument used to prove Theorem 2 (applied to the pair Ω_n, $B_{R_n} + x_0$), we see that, if the overlap is small enough, then there will be a positive solution u_n on $\Omega_n \cup (B_{R_n} + x)$ near (in $L^p(\tilde{B})$) the solution which is 0 on Ω_n and the translate of u_{R_n} on $B_{R_n} + x_0$. Hence $\|u_n\|_\infty \geqslant n - \frac{1}{2}$ (since $\|u_{R_n}\|_\infty \geqslant n$). This gives our required example. Note that we can round off corners to make Ω_n' smooth, that Ω_n' satisfies the basic assumptions of Section 1 (since $R_n \to 0$ as $n \to \infty$), and that Ω_n' will be contractible if Ω_n is.

The above example (and its many variants) shows that, in the superlinear case, the uniform bounds depend on *local* assumptions on the shape of the boundary of Ω_n. (Note that Ω_n' is obtained from Ω_n by small local changes.) We mention one variant of our above example which may be of interest. The uniform bound property fails for a dumbbell with a small bump in it provided the radius of the bump tends to zero sufficiently slowly compared with the width of the joining strip as in Fig. 6.

We want to discuss briefly when the uniform bound holds. Before doing this, we need to discuss where large solutions attain their maxima. Suppose that u_n is a positive solution of (1) (for $\Omega = \Omega_n$ and $f(y) = y^p$) such that $\|u_n\|_\infty \to \infty$ as $n \to \infty$. Suppose $x_n \in \Omega_n$ such that $u_n(x_n) = \|u_n\|_\infty$. Then $\{(\|u_n\|_\infty)^{p-1} d(x_n, \partial\Omega_n)^2\}_{n=1}^\infty$ is bounded. This follows since otherwise we could obtain a contradiction by a slight modification of the blowing up argument in Gidas and Spruck [21]. In particular, x_n must be close to $\partial\Omega_n$. Moreover, by examining the arguments in [21] for blowing up near the boundary, we see that we also get a contradiction if Ω_n is smooth near $y_n \in \partial\Omega_n$, if the maximum of u_n occurs near y_n, and if, near y_n, $\partial\Omega_n$ can be expressed in the form $z = f_n(t)$ where the f_n are uniformly bounded in C^2 and where the size of the neighbourhood of y_n (in $\partial\Omega_n$) is independent of n. Indeed, if we examine the argument in [21] more carefully and if we use barriers, it suffices to have local uniform estimates for the curvature of $\partial\Omega_n$ near y_n. If we do not have a uniform bound, the above result severely restricts where the maximum of u_n can lie.

In some simple but not very interesting cases, the above remarks and the results in Gidas, Ni, and Nirenberg [19] suffice to establish uniform bounds. Consider domains of the type in Fig. 7, where the narrow "snout" has length a where $a < 1$ and the ball has radius 1. If these domains are constructed suitably, Gidas, Ni, and Nirenberg imply that the maximum of

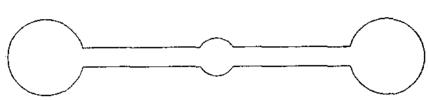

FIGURE 6

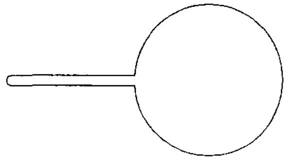

FIGURE 7

a positive solution cannot occur in or near the snout and the results of the previous paragraph then give the result. Note that these domains satisfy the assumptions of Section 1 if the width of the snout tends to zero with n. These are a little uninteresting because they satisfy the assumptions of Section 1 with $s=1$.

More interestingly, consider Ω_n to be the union of two slightly intersecting balls as in Section 1 except that we round off the corner so as to retain the natural $O(m-1)$ symmetry (Z_2 symmetry if $m=2$) and so the width of the smoothed corner is very narrow compared with the width of the gap between the rounded corners (where, in this terminology, we are thinking of the intersection of Ω_n with a natural two-dimensional plane). Moreover, we assume the rounding off is nice as in Fig. 8. In this case, it can be shown the the uniform bound property holds (and thus we have an exact count of the number of positive solutions). We omit the proof here for two reasons. First, it is rather long and tedious and the techniques have nothing to do with the rest of this paper. Second, it seems likely that they can be considerably improved. The idea of the proof is to use a blowing up argument to show that it suffices to prove that there is no nontrivial bounded positive solution of $-\Delta u = u^p$ in $\{x \in R^m : x_1 \neq 0 \text{ or } \|x\| < 1\}$ with $u(x)=0$ when $x_1=0$ and $\|x\|>1$ and with $u=u(x_1, e)$ where $e=\sqrt{x_2^2 \cdots + x_m^2}$. One shows that this is impossible by using a Pokojaev identity argument, by using ideas in [20] and [13] to get good decay

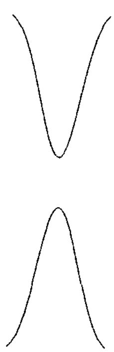

FIGURE 8

estimates at infinity, and by using Grisvard [23] to analyse the behaviour near the singular points. We use the $O(m-1)$ symmetry to ensure we have a two-dimensional problem in order to apply [23]. The same idea still works if we have two balls joined by a tube which is much shorter than it is wide.

The problem in the case of a true dumbbell (that is, a joining strip of finite length but narrow width) is unclear. The difficulty here is the possibility of a solution which is mostly concentrated on the joining strip. For simplicity, assume that $m=2$. It is known that there is a positive solution u_0 of $-\Delta u = u^p$ on the infinite strip T of width 1 with $u_0(x)=0$ on ∂T. Moreover u_0 and its derivatives decay exponentially. These results can be found in Amick and Toland [3]. Let us assume that the strip in Ω_n has length 1 and width n^{-1}. Finally, choose $\psi \in C^2(R)$ such that $\psi(t)=1$ for $|t| \leqslant \frac{1}{4}$, $\psi(t)=0$ for $|t| \geqslant \frac{1}{2}$, and ψ is nonnegative. Then the function $n^{\alpha} u_0(nx)\,\psi(x_1)$ (where $\alpha = 2/(p-1)$) satisfies the boundary conditions on Ω_n and fails to be a solution of the equation on Ω_n by an exponentially small term (by the exponential decay of u_0). This makes what happens in this case seem a little unclear.

6. A Simple Uniqueness Result

In this section, we prove a very simple uniqueness result. It is hoped that it will stimulate more work in the uniqueness question for convex domains.

We consider domains Ω with C^2 boundary in R^2 which contain 0 and which are invariant under the maps $(x, y) \to (-x, y)$ and $(x, y) \to (x, -y)$. (Thus Ω is determined by its intersection with the first quadrant.) Moreover, we assume that the part of Ω in the first quadrant is given by $\{(x, y): 0 \leqq y \leqq h(x):\ 0 \leqq x \leqq a\}$ where h is a decreasing function with $h(a)=0$.

Theorem 5. *If Ω is as above and $1 < p < (m+2)/(m-2)$, then the equation*

$$\begin{aligned} -\Delta u &= u^p \quad \text{in } \Omega \\ u &= 0 \quad \text{on } \partial\Omega \end{aligned} \tag{25}$$

has a unique positive solution and this solution is nondegenerate.

Remark. This result was announced in [13].

Proof. It suffices to prove that any positive solution is nondegenerate for Ω as above. To see this, note that it is easy to construct a C^2 deformation of Ω to the unit ball such that each intermediate domain $\Omega(t)$

satisfies the same assumptions as Ω. By the remarks in Section 5, we have a uniform bound for positive solutions on the $\Omega(t)$'s. (In fact, since Gidas, Ni, and Nirenberg ensure that the maximum of a positive solution occurs at zero, we do not have to worry about boundary blow-ups). Hence, by the proof of Theorem 2, we see that the number of positive solutions on $\Omega(t)$ is locally constant in t. Note that we really use a much weaker result than Theorem 2 and that we use the invertibility result here. Since there is a unique positive solution on a ball, the result follows.

We still have to prove the invertibility condition. By Gidas, Ni, and Nirenberg, any positive solution u is even in x and y. It follows easily that, if there is a nontrivial solution of the linearized equation

$$\begin{aligned} -\Delta h &= pu^{p-1}h && \text{in } \Omega \\ h &= 0 && \text{on } \partial\Omega, \end{aligned} \tag{26}$$

then there is one which is (i) odd in x and y, (ii) odd in x and even in y, (iii) even in x and odd in y, or (iv) even in x and y. This is really a consequence of the symmetries.

By differentiating (25) with respect to x and y, we see that the partial derivatives u_x and u_y satisfy the linearized equation (26) but *not* the boundary condition.

We complete the proof in 2 steps.

Step 1. There is no nontrivial solution of (26) of the type (i) or (ii) or (iii). We consider (ii). The others are similar or easier. We consider Ω_1, the part of Ω in the interior of the first quadrant. Suppose h is a nontrivial solution of (26) of the type (ii). Then, on $\partial\Omega_1$, h is zero on the curved part and on the vertical part (by the oddness in x) and $h_y = 0$ when $y = 0$. Now $w = u_x$ satisfies the same equation as h. Suppose that T_2 is a component of $\{x \in \Omega_1 : h(x) \neq 0\}$. On ∂T_2, h will be zero except on the horizontal part of the boundary. If we multiply the equation for h by w, integrate over T_2, and use Green's theorem, we see that

$$\int_{T_3} \frac{\partial h}{\partial n} w = 0, \tag{27}$$

where T_3 is the part of ∂T_2 not on the axes. Note that, by the parity conditions on u, $w = 0$ when $x = 0$ and $w_y = 0$ on $y = 0$. Without loss of generality, $h > 0$ on T_2. By the maximum principle, $(\partial h/\partial n)(x) < 0$ on T_3 except at the corners of T_2. Since $w > 0$ on T_3 (by [19]) we see that (27) is impossible. Note that T_3 is nonempty. One point should be made here. The result of Hartman and Wintner [25] ensures that T_2 is smooth enough to apply Green's theorem.

If h is of type (i), we can use the same argument except that in (27) T_3 should be replaced by ∂T_2. In this case, w is nonnegative on ∂T_2 and positive on an open subset of ∂T_2.

Step 2. We prove that (26) cannot have a nontrivial solution of type (iv). We use the special form of the nonlinearity more here. Note that if u is a solution of (26) (possibly not satisfying the boundary conditions), then $\alpha^{2/(p-1)}u(\alpha x)$ is a solution for all α positive. In particular, by differentiating this expression in α at $\alpha = 1$, we see that $v(x) = 2/(p-1)\, u(x) + x \cdot \nabla u(x)$ is a solution of (26) (*but not the boundary condition*). We use this and u_x and u_y to get a contradiction. By Gidas, Ni, and Nirenberg, $v < 0$ on the curved part of $\partial\Omega_1$. If $y = 0$,

$$\begin{aligned} v_y &= 2/(p-1)\, u_y + x u_{xy} + u_y \\ &= 0 \end{aligned}$$

by our parity assumptions. Similarly $v_x = 0$ on $x = 0$. Thus v satisfies the same boundary conditions as h on the straight part of $\partial\Omega_1$. Hence much as in part (i) we can multiply the equation for h by v, integrate over Ω_1, and use Green's theorem to deduce that $\int_{T_1} (\partial h/\partial n)\, v = 0$. Here T_1 is the curved part of $\partial\Omega_1$. Since $v > 0$ on T_1, $\partial h/\partial n$ must change sign on T_1 and hence h must have interior nodal lines meeting T_1. It follows easily (and here we use that we are in two dimensions) that there is a component T_4 of $\{x \in \Omega_1 : h(x) \neq 0\}$ not intersecting either $x = 0$ or $y = 0$. Without loss of generality, we assume that T_4 does not intersect $y = 0$ and that $h > 0$ on T_4. On $x = 0$, $z = u_x$ satisfies the same boundary conditions as h while on the rest of ∂T_4, h is zero. Hence much as before, we find that $\int_{T_5} (\partial h/\partial n)\, z = 0$, where T_5 is the part of T_4 not on $x = 0$. Now on T_5, $\partial h/\partial n < 0$ except at the corners and $z < 0$ (by [19]). Hence we have a contradiction. This completes the proof.

Note that Step 1 is unnecessary if we only want to prove the uniqueness. We only use the special form of the nonlinearity to eliminate h's of type (iv) with the property that every component of $\{x \in \Omega_1 : h(x) \neq 0\}$ intersects both $x = 0$ and $y = 0$. The analogue of Step 1 of the proof seems to work in all dimensions (for suitable domains). The only problem is in justifying the use of Green's theorem. Note that our proof of Step 2 uses essentially that we are in two dimensions. However, it seems likely that the proof can be generalized to cover m-dimensional domains with an $O(m-2)$ symmetry (as well as the m-dimensional analogue of our conditions above). To improve our result to three dimensions, we have to show that certain rather odd typed h's cannot occur. (Every nodal domain in the first quadrant would have to intersect the three flat surfaces and the curved surface.)

It is unclear if our methods can be generalized to apply to the Gelfand equation. Finally, note that our ideas imply that uniqueness holds in Theorem 5 for domains C^2 close to the ones in Theorem 5.

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